

## ON COMPLETIONS OF HECKE ALGEBRAS

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ABSTRACT. Let  $G$  be a reductive  $p$ -adic group and let  $\mathcal{H}(G)^s$  be a Bernstein block of the Hecke algebra of  $G$ . We consider two important topological completions of  $\mathcal{H}(G)^s$ : a direct summand  $\mathcal{S}(G)^s$  of the Harish-Chandra–Schwartz algebra of  $G$  and a two-sided ideal  $C_r^*(G)^s$  of the reduced  $C^*$ -algebra of  $G$ . These are useful for the study of all tempered smooth  $G$ -representations.

We suppose that  $\mathcal{H}(G)^s$  is Morita equivalent to an affine Hecke algebra  $\mathcal{H}(\mathcal{R}, q)$  – as is known in many cases. The latter algebra also has a Schwartz completion  $\mathcal{S}(\mathcal{R}, q)$  and a  $C^*$ -completion  $C_r^*(\mathcal{R}, q)$ , both defined in terms of the underlying root datum  $\mathcal{R}$  and the parameters  $q$ .

We prove that, under some mild conditions, a Morita equivalence  $\mathcal{H}(G)^s \sim_M \mathcal{H}(\mathcal{R}, q)$  extends to Morita equivalences  $\mathcal{S}(G)^s \sim_M \mathcal{S}(\mathcal{R}, q)$  and  $C_r^*(G)^s \sim_M C_r^*(\mathcal{R}, q)$ . We also check that our conditions are fulfilled in all known cases of such Morita equivalences between Hecke algebras.

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## INTRODUCTION

Let  $G$  be a connected reductive group over a local non-archimedean field. Let  $\text{Rep}(G)$  be the category of smooth  $G$ -representations on complex vector spaces. To study such representations, it is often useful to consider various group algebras of  $G$ . Most fundamentally, there is the Hecke algebra  $\mathcal{H}(G)$ , the convolution algebra of locally constant, compactly supported functions  $G \rightarrow \mathbb{C}$ . The category  $\text{Rep}(G)$  is equivalent to the category  $\text{Mod}(\mathcal{H}(G))$  of nondegenerate  $\mathcal{H}(G)$ -modules. (Here  $V$  nondegenerate means that  $\mathcal{H}(G) \cdot V = V$ .)

For purposes of harmonic analysis, and in particular for the study of tempered smooth  $G$ -representations, the Harish-Chandra–Schwartz algebra  $\mathcal{S}(G)$  [HC] can

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be very convenient. This is a topological completion of  $\mathcal{H}(G)$ , it consists of locally constant functions  $G \rightarrow \mathbb{C}$  that decay rapidly in a specified sense. By [Wal, §III.7] an admissible smooth  $G$ -representation is tempered if and only if it is naturally a  $\mathcal{S}(G)$ -module. For larger representations it is best to define the category of tempered smooth  $G$ -representations as the category  $\text{Mod}(\mathcal{S}(G))$  of nondegenerate  $\mathcal{S}(G)$ -modules [SSZ, Appendix].

Finally, from the point of view of operator algebras or noncommutative geometry, the reduced  $C^*$ -algebra  $C_r^*(G)$  may be the most suitable. The modules of  $C_r^*(G)$  are Hilbert spaces, so they are usually not smooth as  $G$ -representations. But  $\mathcal{S}(G) \subset C_r^*(G)$  and the smooth vectors in any  $C_r^*(G)$ -module do form a  $\mathcal{S}(G)$ -module and hence a  $\mathcal{H}(G)$ -module. Moreover this operation provides a bijection between the irreducible representations of  $C_r^*(G)$  and those of  $\mathcal{S}(G)$ . This feature distinguishes  $C_r^*(G)$  from other Banach group algebras like  $L^1(G)$  or the maximal  $C^*$ -algebra of  $G$ .

Let  $\text{Rep}(G)^\mathfrak{s}$  be a Bernstein block of  $\text{Rep}(G)$  [BeDe]. It is well-known that in many cases (see Sections 4 and 5)  $\text{Rep}(G)^\mathfrak{s}$  is equivalent to the category of modules of an affine Hecke algebra  $\mathcal{H}(\mathcal{R}, q)$ . Here  $\mathcal{R}$  is a root datum and  $q$  is a parameter function for  $\mathcal{R}$ . In such cases it would be useful if one could detect, in terms of  $\mathcal{H}(\mathcal{R}, q)$ , whether a  $G$ -representation in  $\text{Rep}(G)^\mathfrak{s}$

- (i) is tempered;
- (ii) is unitary;
- (iii) admits a continuous extension to a  $C_r^*(G)$ -module.

The structure needed to make sense of this is available for (extended) affine Hecke algebras with positive parameters. They have a natural  $*$ -operation, so unitarity is defined. Temperedness of finite-dimensional  $\mathcal{H}(\mathcal{R}, q)$ -modules can be defined conveniently either in terms of growth of matrix coefficient or by means of weights for a large commutative subalgebra of  $\mathcal{H}(\mathcal{R}, q)$  [Opd, §2.7].

Furthermore there exists a Schwartz completion  $\mathcal{S}(\mathcal{R}, q)$  of  $\mathcal{H}(\mathcal{R}, q)$  [Opd, §6.2] with a similar structure as  $\mathcal{S}(G)$  [DeOp]. By [Opd, Corollary 6.7] a finite dimensional  $\mathcal{H}(\mathcal{R}, q)$ -module is tempered if and only if it extends continuously to a  $\mathcal{S}(\mathcal{R}, q)$ -module. Like for  $G$ , we define the category of tempered  $\mathcal{H}(\mathcal{R}, q)$ -modules to be the module category of  $\mathcal{S}(\mathcal{R}, q)$ .

The algebra  $\mathcal{H}(\mathcal{R}, q)$  is a Hilbert algebra, so it acts by multiplication on its own Hilbert space completion. Then one can define the reduced  $C^*$ -completion  $C_r^*(\mathcal{R}, q)$  as the closure of  $\mathcal{H}(\mathcal{R}, q)$  in the algebra of bounded linear operators on that Hilbert space. It is reasonable to expect that this algebra plays a role analogous to  $C_r^*(G)$ .

Let  $\mathcal{H}(G)^\mathfrak{s}$  (resp.  $\mathcal{S}(G)^\mathfrak{s}$  and  $C_r^*(G)^\mathfrak{s}$ ) be the direct summand of  $\mathcal{H}(G)$  (resp.  $\mathcal{S}(G)$  and  $C_r^*(G)$ ) corresponding to  $\text{Rep}(G)^\mathfrak{s}$  via the Bernstein decomposition

$$\text{Mod}(\mathcal{H}(G)) \cong \text{Rep}(G) = \prod_{\mathfrak{s} \in \mathfrak{B}(G)} \text{Rep}(G)^\mathfrak{s}.$$

In view of the above, it is natural to ask whether an equivalence of categories

$$(1) \quad \text{Rep}(G)^\mathfrak{s} \cong \text{Mod}(\mathcal{H}(G)^\mathfrak{s}) \xrightarrow{\sim} \text{Mod}(\mathcal{H}(\mathcal{R}, q))$$

extends to Morita equivalences

$$(2) \quad \mathcal{S}(G)^\mathfrak{s} \sim_M \mathcal{S}(\mathcal{R}, q) \quad \text{and} \quad C_r^*(G)^\mathfrak{s} \sim_M C_r^*(\mathcal{R}, q).$$

That would solve the issues (i) and (iii) completely, and would provide a partial answer to (ii). Namely, it would imply that (1) matches the unitary tempered representations on both sides. (It is not clear what it could say about unitary non-tempered representations.) Furthermore (2) would make  $\text{Mod}(\mathcal{S}(G)^\natural)$  and  $\text{Mod}(C_r^*(G)^\natural)$  amenable to much more explicit calculations, in terms of the generators and relations from  $\mathcal{H}(\mathcal{R}, q)$ .

While (2) looks fairly plausible, it is not automatic. To prove it, we impose some conditions on the Morita equivalence  $\mathcal{H}(G)^\natural \sim_M \mathcal{H}(\mathcal{R}, q)$ :

- Condition 3.1 is about compatibility with parabolic induction and restriction.
- Condition 3.2 says roughly that under this Morita equivalence every (suitable) parabolic subgroup of  $G$  should give rise to a parabolic subalgebra of  $\mathcal{H}(\mathcal{R}, q)$ , and this correspondence should respect positivity in the underlying root systems.
- Sometimes we obtain, instead of  $\mathcal{H}(\mathcal{R}, q)$ , an extended affine Hecke algebra  $\mathcal{H}(\mathcal{R}, q) \rtimes \Gamma$ , where  $\Gamma$  is a finite group. Then we require Condition 1.2, which says that  $\Gamma$  respects all the relevant structure.

**Theorem 1.** (see Theorem 3.4)

*Suppose that  $\mathcal{H}(G)^\natural$  is Morita equivalent with an extended affine Hecke algebra  $\mathcal{H}(\mathcal{R}, q) \rtimes \Gamma$  with positive parameters, such that Conditions 3.1, 3.2 and 1.2 hold. Then it induces Morita equivalences  $\mathcal{S}(G)^\natural \sim_M \mathcal{S}(\mathcal{R}, q)$  and  $C_r^*(G)^\natural \sim_M C_r^*(G)$ .*

Hitherto this was only proven for the Schwartz completions in the case of Iwahori-spherical representations of split groups [DeOp, Theorem 10.2]. In all cases where a Morita equivalence on the Hecke algebra level is known (to the author), we check that the conditions of Theorem 1 are fulfilled. This includes principal series representations of  $F$ -split groups ( $F$  is any local non-archimedean field), level zero representations, inner forms of  $\text{GL}_n(F)$ , inner forms of  $\text{SL}_n(F)$ , symplectic groups (not necessarily split) and special orthogonal groups (also possibly non-split).

In all these cases we obtain a pretty good picture of  $C_r^*(G)^\natural$ , up to Morita equivalence. This can, for instance, be used to compute the topological K-theory of these algebras. Indeed, in [Sol2] the author determined the K-theory of  $C_r^*(\mathcal{R}, q)$  for many root data  $\mathcal{R}$  (it does not depend on  $q$ ). These calculations, together with Theorem 1 for classical groups, lead to a result which is useful in relation with the Baum–Connes conjecture.

**Theorem 2.** (see Theorem 5.3)

*Let  $G$  be a special orthogonal or a symplectic group over  $F$  (not necessarily split), or an inner form of  $\text{GL}_n(F)$ . Then  $K_*(C_r^*(G))$  is a free abelian group. For every Bernstein block  $\text{Rep}(G)^\natural$ , the rank of  $K_*(C_r(G)^\natural)$  is finite and can be computed explicitly.*

Let us also discuss other approaches to the issues (i), (ii) and (iii). In most cases where a Morita equivalence  $\mathcal{H}(G)^\natural \sim_M \mathcal{H}(\mathcal{R}, q) \rtimes \Gamma$  is known, these issues are not discussed in the literature. When the Morita equivalence comes from a type  $(K, \lambda)$  in the sense of Bushnell–Kutzko [BuKu], some techniques are available. In this setting the Morita equivalence can be implemented by an injective algebra homomorphism

$$(3) \quad \Upsilon_\lambda : \mathcal{H}(\mathcal{R}, q) \rtimes \Gamma \rightarrow \mathcal{H}(G)^\natural,$$

see [BuKu, (2.12)]. If  $\Upsilon_\lambda$  is a  $*$ -homomorphism, then the Morita equivalence matches the Hermitian modules on both sides [BaMo, §2]. Unitary  $\mathcal{H}(G)^\natural$ -modules are sent to unitary  $\mathcal{H}(\mathcal{R}, q) \rtimes \Gamma$ -modules, but maybe not conversely [Bor, §5.11].

Both  $\mathcal{H}(G)$  and  $\mathcal{H}(\mathcal{R}, q) \rtimes \Gamma$  are endowed with a canonical trace, which stems from evaluation of functions at the unit element of  $G$ . Usually  $\Upsilon_\lambda$  will transfer the trace on  $\mathcal{H}(G)^\natural$  to a positive scalar multiple of the trace on  $\mathcal{H}(\mathcal{R}, q) \rtimes \Gamma$ . If that is the case and  $\Upsilon_\lambda$  is a  $*$ -homomorphism, then by [DeOp, Theorem 10.1] (3) induces an equivalence between the category of finite length tempered  $G$ -representations in  $\text{Rep}(G)^\natural$  and the category of finite dimensional tempered modules of  $\mathcal{H}(\mathcal{R}, q) \rtimes \Gamma$ . This relies on properties of the Plancherel measures of  $G$  and of  $\mathcal{H}(\mathcal{R}, q) \rtimes \Gamma$ , and it uses that  $\Upsilon_\lambda$  preserves these Plancherel measures, up to a scalar multiple.

Let  $e_\natural \in \mathcal{H}(G)$  be the idempotent associated to the type  $(K, \lambda)$ . Then the algebra  $e_\natural \mathcal{H}(G) e_\natural$  lies inbetween  $\mathcal{H}(\mathcal{R}, q) \rtimes \Gamma$  and  $\mathcal{H}(G)$ . The inclusion  $e_\natural \mathcal{H}(G) e_\natural \rightarrow \mathcal{H}(G)$  always preserves all the aforementioned properties [BHK], this follows from general principles for Hilbert algebras and does not rely on the finer structure of the algebras. So the difficult part of (3) is the map  $\mathcal{H}(\mathcal{R}, q) \rtimes \Gamma \rightarrow e_\natural \mathcal{H}(G) e_\natural$ , where the left hand side is considered with its structure as affine Hecke algebra and the right hand side as subalgebra of  $\mathcal{H}(G)$ .

Suppose that, on top of the above,  $\Upsilon_\lambda$  satisfies the conditions in [BaCi, Definition 5.4.1]. Then (3) induces a bijection between the unitary duals of  $\mathcal{H}(G)^\natural$  and  $\mathcal{H}(\mathcal{R}, q) \rtimes \Gamma$  [BaCi, Theorem 5.5.2]. These extra conditions seems rather mild, Barbasch and Ciubotaru check them in many cases.

When the Morita equivalence  $\mathcal{H}(G)^\natural \sim_M \mathcal{H}(\mathcal{R}, q) \rtimes \Gamma$  does not arise from a type, fewer techniques for (i), (ii) and (iii) were known. Heiermann established such Morita equivalences, for symplectic and special orthogonal groups and for inner forms of  $\text{GL}_n(F)$  [Hei1], by using Bernstein's progenerators of  $\text{Rep}(G)^\natural$ . In [Hei2] he showed that these equivalences preserve temperedness of finite length modules. Here it is unknown whether the  $*$  and the trace are preserved by the Morita equivalence. Since a map like (3) is lacking, it is even unclear how such a statement could be formulated in this setting.

Summarizing, in the literature already several results about the behaviour of finite length modules under a Morita equivalence between a Bernstein block  $\text{Rep}(G)^\natural$  and the module category of an (extended) affine Hecke algebra can be found, but there is so far almost nothing about the Schwartz completions and the  $C^*$ -completions.

Let us briefly describe the contents of the paper. In the first section we recall the definitions of affine Hecke algebras and their topological completions. We formulate the Plancherel isomorphism for these completions, from [DeOp], and we establish suitable versions for affine Hecke algebras extended with finite groups.

After that we look at the aforementioned group algebras for a reductive  $p$ -adic group  $G$ . We recall the Plancherel isomorphism for the Schwartz algebra of  $G$  [HC, Wal] and for the reduced  $C^*$ -algebra of  $G$  [Ply].

This forms the setup for the proof of our main result Theorem 1, which occupies Section 3. The crucial idea behind our argument is that in the Plancherel isomorphisms for  $\mathcal{S}(G)^\natural$  and  $\mathcal{S}(\mathcal{R}, q)$  very similar algebras arise. In both settings one encounters a bundle of matrix algebras over a compact torus, one takes  $C^\infty$ -sections of those, and then invariants with respect to a finite group acting via intertwining

operators. We compare the resulting algebras on both sides, analysing the data used to describe the Plancherel isomorphisms. More specifically, we prove that a Morita equivalence between  $\mathcal{H}(G)^s$  and  $\mathcal{H}(\mathcal{R}, q)$ , plus some mild extra conditions, implies that the two necessary sets of data, for  $\mathcal{S}(G)^s$  and for  $\mathcal{S}(\mathcal{R}, q)$ , become equivalent after some manipulations. This gives us the desired Morita equivalences between topological algebras.

In Section 4 we check that the conditions from Section 3 are fulfilled in (most) known cases of Morita equivalences coming from types. In the final section 5 we do the same for Heiermann's Morita equivalences constructed with the use of projective generators, and we derive Theorem 2.

## 1. AFFINE HECKE ALGEBRAS AND THEIR COMPLETIONS

Let  $\mathfrak{a}$  be a finite dimensional real vector space and let  $\mathfrak{a}^*$  be its dual. Let  $Y \subset \mathfrak{a}$  be a lattice and  $X = \text{Hom}_{\mathbb{Z}}(Y, \mathbb{Z}) \subset \mathfrak{a}^*$  the dual lattice. Let

$$(4) \quad \mathcal{R} = (X, R, Y, R^\vee, \Delta).$$

be a based root datum. Thus  $R$  is a reduced root system in  $X$ ,  $R^\vee \subset Y$  is the dual root system,  $\Delta$  is a basis of  $R$  and the set of positive roots is denoted  $R^+$ . Furthermore we are given a bijection  $R \rightarrow R^\vee$ ,  $\alpha \mapsto \alpha^\vee$  such that  $\langle \alpha, \alpha^\vee \rangle = 2$  and such that the corresponding reflections  $s_\alpha : X \rightarrow X$  (resp.  $s_\alpha^\vee : Y \rightarrow Y$ ) stabilize  $R$  (resp.  $R^\vee$ ). We do not assume that  $R$  spans  $\mathfrak{a}^*$ . The reflections  $s_\alpha$  generate the Weyl group  $W = W(R)$  of  $R$ , and  $S_\Delta := \{s_\alpha \mid \alpha \in \Delta\}$  is the collection of simple reflections.

We have the affine Weyl group  $W^{\text{aff}} = \mathbb{Z}R \rtimes W$  and the extended (affine) Weyl group  $W^e = X \rtimes W$ . Both can be considered as groups of affine transformations of  $\mathfrak{a}^*$ . We denote the translation corresponding to  $x \in X$  by  $t_x$ . As is well-known,  $W^{\text{aff}}$  is a Coxeter group, and the basis  $\Delta$  of  $R$  gives rise to a set  $S^{\text{aff}}$  of simple (affine) reflections. More explicitly, let  $\Delta_M^\vee$  be the set of maximal elements of  $R^\vee$ , with respect to the dominance ordering coming from  $\Delta$ . Then

$$S^{\text{aff}} = S_\Delta \cup \{t_\alpha s_\alpha \mid \alpha^\vee \in \Delta_M^\vee\}.$$

The length function  $\ell$  of the Coxeter system  $(W^{\text{aff}}, S^{\text{aff}})$  extends naturally to  $W^e$ . The elements of length zero form a subgroup  $\Omega \subset W^e$  and  $W^e = W^{\text{aff}} \rtimes \Omega$ .

A complex parameter function for  $\mathcal{R}$  is a map  $q : S^{\text{aff}} \rightarrow \mathbb{C}^\times$  such that  $q(s) = q(s')$  if  $s$  and  $s'$  are conjugate in  $W^e$ . This extends naturally to a map  $q : W^e \rightarrow \mathbb{C}^\times$  which is 1 on  $\Omega$  and satisfies

$$q(ww') = q(w)q(w') \quad \text{if} \quad \ell(ww') = \ell(w) + \ell(w').$$

Equivalently (see [Lus1, §3.1]) one can define  $q$  as a  $W$ -invariant function

$$(5) \quad q : R \cup \{2\alpha : \alpha^\vee \in 2Y\} \rightarrow \mathbb{C}^\times.$$

We speak of equal parameters if  $q(s) = q(s') \forall s, s' \in S^{\text{aff}}$  and of positive parameters if  $q(s) \in \mathbb{R}_{>0} \forall s \in S^{\text{aff}}$ . We fix a square root  $q^{1/2} : S^{\text{aff}} \rightarrow \mathbb{C}^\times$ .

The affine Hecke algebra  $\mathcal{H} = \mathcal{H}(\mathcal{R}, q)$  is the unique associative complex algebra with basis  $\{N_w \mid w \in W^e\}$  and multiplication rules

$$(6) \quad \begin{aligned} N_w N_{w'} &= N_{ww'} && \text{if } \ell(ww') = \ell(w) + \ell(w'), \\ (N_s - q(s)^{1/2})(N_s + q(s)^{-1/2}) &= 0 && \text{if } s \in S^{\text{aff}}. \end{aligned}$$

In the literature one also finds this algebra defined in terms of the elements  $q(s)^{1/2}N_s$ , in which case the multiplication can be described without square roots. This explains why  $q^{1/2}$  does not appear in the notation  $\mathcal{H}(\mathcal{R}, q)$ . For  $q = 1$  (6) just reflects the defining relations of  $W^e$ , so  $\mathcal{H}(\mathcal{R}, 1) = \mathbb{C}[W^e]$ .

The set of dominant elements in  $X$  is

$$X^+ = \{x \in X : \langle x, \alpha^\vee \rangle \geq 0 \forall \alpha \in \Delta\}.$$

The subset  $\{N_{t_x} : x \in X^+\} \subset \mathcal{H}(\mathcal{R}, q)$  is closed under multiplication, and isomorphic to  $X^+$  as a semigroup. For any  $x \in X$  we put

$$\theta_x = N_{t_{x_1}} N_{t_{x_2}}^{-1}, \text{ where } x_1, x_2 \in X^+ \text{ and } x = x_1 - x_2.$$

This does not depend on the choice of  $x_1$  and  $x_2$ , so  $\theta_x \in \mathcal{H}(\mathcal{R}, q)^\times$  is well-defined. The Bernstein presentation of  $\mathcal{H}(\mathcal{R}, q)$  [Lus1, §3] says that:

- $\{\theta_x : x \in X\}$  forms a  $\mathbb{C}$ -basis of a subalgebra of  $\mathcal{H}(\mathcal{R}, q)$  isomorphic to  $\mathbb{C}[X] \cong \mathcal{O}(T)$ , which we identify with  $\mathcal{O}(T)$ .
- $\mathcal{H}(W, q) := \mathbb{C}\{N_w : w \in W\}$  is a finite dimensional subalgebra of  $\mathcal{H}(\mathcal{R}, q)$  (known as the Iwahori–Hecke algebra of  $W$ ).
- The multiplication map  $\mathcal{O}(T) \otimes \mathcal{H}(W, q) \rightarrow \mathcal{H}(\mathcal{R}, q)$  is a  $\mathbb{C}$ -linear bijection.
- There are explicit cross relations between  $\mathcal{H}(W, q)$  and  $\mathcal{O}(T)$ , deformations of the standard action of  $W$  on  $\mathcal{O}(T)$ .

To define parabolic subalgebras of affine Hecke algebras, we associate some objects to any set of simple roots  $Q \subset \Delta$ . Let  $R_Q$  be the root system they generate,  $R_Q^\vee$  the root system generated by  $Q^\vee$  and  $W_Q$  their Weyl group. We also define

$$\begin{aligned} X_Q &= X / (X \cap (Q^\vee)^\perp) & X^Q &= X / (X \cap QQ), \\ Y_Q &= Y \cap QQ^\vee & Y^Q &= Y \cap Q^\perp, \\ T_Q &= \text{Hom}_{\mathbb{Z}}(X_Q, \mathbb{C}^\times) & T^Q &= \text{Hom}_{\mathbb{Z}}(X^Q, \mathbb{C}^\times), \\ \mathcal{R}_Q &= (X_Q, R_Q, Y_Q, R_Q^\vee, Q) & \mathcal{R}^Q &= (X, R_Q, Y, R_Q^\vee, Q), \\ \mathcal{H}_Q &= \mathcal{H}(\mathcal{R}_Q, q_Q) & \mathcal{H}^Q &= \mathcal{H}(\mathcal{R}^Q, q^Q). \end{aligned}$$

Here  $q_Q$  and  $q^Q$  are derived from  $q$  via (5). Both  $\mathcal{H}_Q$  and  $\mathcal{H}^Q$  are called parabolic subalgebras of  $\mathcal{H}$ . The quotient map  $X \mapsto X_Q$  yields a natural projection

$$(7) \quad \mathcal{H}^Q \rightarrow \mathcal{H}_Q : \theta_x N_w \mapsto \theta_{x_Q} N_w.$$

In this way one can regard  $\mathcal{H}_Q$  as a “semisimple” quotient of  $\mathcal{H}^Q$ . The algebra  $\mathcal{H}^Q$  is embedded in  $\mathcal{H}$  via the Bernstein presentation, as the image of  $\mathcal{O}(T) \otimes \mathcal{H}(W_Q, q) \rightarrow \mathcal{H}$ . Any  $t \in T^Q$  and any  $u \in T^Q \cap T_Q$  give rise to algebra automorphisms

$$(8) \quad \begin{aligned} \psi_u : \mathcal{H}_Q &\rightarrow \mathcal{H}_Q, & \theta_{x_Q} N_w &\mapsto u(x_Q) \theta_{x_Q} N_w, \\ \psi_t : \mathcal{H}^Q &\rightarrow \mathcal{H}^Q, & \theta_x N_w &\mapsto t(x) \theta_x N_w. \end{aligned}$$

Let  $\Gamma$  be a finite group acting on  $\mathcal{R}$ , i.e. it acts  $\mathbb{Z}$ -linearly on  $X$  and preserves  $R$  and  $\Delta$ . We also assume that  $\Gamma$  acts on  $T$  by affine transformations, whose linear part comes from the action on  $X$ . Thus  $\Gamma$  acts on  $\mathcal{O}(T) \cong \mathbb{C}[X]$  by

$$(9) \quad \gamma(\theta_x) = z_\gamma(x) \theta_{\gamma x},$$

for some  $z_\gamma \in T$ . Since this is a group action, we must have  $z_\gamma \in T^W$ .

We suppose throughout that  $q^{1/2}$  is  $\Gamma$ -invariant, so that  $\gamma \in \Gamma$  acts on  $\mathcal{H}(\mathcal{R}, q)$  by the algebra automorphism

$$(10) \quad \sum_{w \in W, x \in X} c_{x,w} \theta_x N_w \mapsto \sum_{w \in W, x \in X} c_{x,w} z_\gamma(x) \theta_{\gamma(x)} N_{\gamma w \gamma^{-1}}.$$

We can build the crossed product algebra

$$(11) \quad \mathcal{H}(\mathcal{R}, q) \rtimes \Gamma.$$

In [Sol1] we considered a slightly less general action of  $\Gamma$  on  $\mathcal{H}(\mathcal{R}, q)$ , where the elements  $z_\gamma \in T^W$  from (9) were all equal to 1. But the relevant results from [Sol1] do not rely on  $\Gamma$  fixing the unit element of  $T$ , so they are also valid for the actions as in (10). In this paper we will tacitly use some results from [Sol1] in the generality of (10). We note that nontrivial  $z_\gamma \in T^W$  are sometimes needed to describe Hecke algebras coming from  $p$ -adic groups, for example in [Roc, §4].

It follows from the Bernstein presentation that the centre of  $\mathcal{H}(\mathcal{R}, q) \rtimes \Gamma$  contains  $\mathcal{O}(T)^{W\Gamma} = \mathcal{O}(T/W\Gamma)$ , with equality if  $\Gamma$  acts faithfully on  $T$ . By Schur's Lemma  $\mathcal{O}(T)^{W\Gamma}$  acts on every irreducible  $\mathcal{H} \rtimes \Gamma$ -representation  $\pi$  by a character. Such a character can be identified with a  $W\Gamma$ -orbit  $W\Gamma t \subset T$ . We will just call  $W\Gamma t$  the central character of  $\pi$ .

Since  $\mathcal{H}(\mathcal{R}, q)$  is of finite rank as a module over its commutative subalgebra  $\mathcal{O}(T)$ , all irreducible  $\mathcal{H}(\mathcal{R}, q)$ -modules have finite dimension. The set of  $\mathcal{O}(T)$ -weights of a  $\mathcal{H}(\mathcal{R}, q)$ -module  $V$  will be denoted by  $\text{Wt}(V)$ .

We regard  $\mathfrak{t} = \mathfrak{a} \oplus i\mathfrak{a}$  as the polar decomposition of  $\mathfrak{t}$ , with associated real part map  $\Re : \mathfrak{t} \rightarrow \mathfrak{a}$ . The vector space  $\mathfrak{t}$  can be interpreted as the Lie algebra of the complex torus  $T = \text{Hom}_{\mathbb{Z}}(X, \mathbb{C}^\times)$ . The latter has a polar decomposition  $T = T_{\text{rs}} \times T_{\text{un}}$  where  $T_{\text{rs}} = \text{Hom}_{\mathbb{Z}}(X, \mathbb{R}_{>0})$  and  $T_{\text{un}}$  is the unique maximal compact subgroup of  $T$ . The polar decomposition of an element  $t \in T$  is written as  $t = |t| (t|t|^{-1})$ . We write

$$\begin{aligned} \mathfrak{a}^+ &= \{\mu \in \mathfrak{a} : \langle \alpha, \mu \rangle \geq 0 \forall \alpha \in \Delta\}, \\ \mathfrak{a}^{*+} &:= \{x \in \mathfrak{a}^* : \langle x, \alpha^\vee \rangle \geq 0 \forall \alpha \in \Delta\}, \\ \mathfrak{a}^- &= \{\lambda \in \mathfrak{a} : \langle x, \lambda \rangle \leq 0 \forall x \in \mathfrak{a}^{*+}\} = \left\{ \sum_{\alpha \in \Delta} \lambda_\alpha \alpha^\vee : \lambda_\alpha \leq 0 \right\}. \end{aligned}$$

The interior  $\mathfrak{a}^{--}$  of  $\mathfrak{a}^-$  equals  $\{\sum_{\alpha \in \Delta} \lambda_\alpha \alpha^\vee : \lambda_\alpha < 0\}$  if  $\Delta$  spans  $\mathfrak{a}^*$ , and is empty otherwise. We write

$$T^- = \exp(\mathfrak{a}^-) \subset T_{\text{rs}} \quad \text{and} \quad T^{--} = \exp(\mathfrak{a}^{--}) \subset T_{\text{rs}}.$$

We say that a module  $V$  for  $\mathcal{H}(\mathcal{R}, q)$  (or for  $\mathcal{H}(\mathcal{R}, q) \rtimes \Gamma$ ) is tempered if  $|\text{Wt}(V)| \subset T^-$ , and that it is discrete series if  $|\text{Wt}(V)| \subset T^{--}$ . The latter is only possible if  $R$  spans  $\mathfrak{a}$ , for otherwise  $\mathfrak{a}^{--}$  and  $T^{--}$  are empty. We alleviate this notion by calling a  $\mathcal{H} \rtimes \Gamma$ -module essentially discrete series if its restriction to  $\mathcal{H}_\Delta$  is discrete series. This means that  $\text{Wt}(V) \subset T^{--} T_{\text{un}} T^\Delta$ . We denote the set of (equivalence classes of) irreducible essentially discrete series representations by  $\text{Irr}_{L^2}(\mathcal{H}(\mathcal{R}, q) \rtimes \Gamma)$ .

To get nice completions of  $\mathcal{H}(\mathcal{R}, q)$  we assume from now on that  $q$  is a positive parameter function for  $\mathcal{R}$ . As a topological vector space the Schwartz completion of  $\mathcal{H}(\mathcal{R}, q)$  will consist of rapidly decreasing functions on  $W^e$ , with respect to a suitable length function  $\mathcal{N}$ . For example we can take a  $W$ -invariant norm on  $X \otimes_{\mathbb{Z}} \mathbb{R}$  and

put  $\mathcal{N}(wt_x) = \|x\|$  for  $w \in W$  and  $x \in X$ . Then we can define, for  $n \in \mathbb{N}$ , the following norm on  $\mathcal{H}$ :

$$p_n\left(\sum_{w \in W^e} h_w N_w\right) = \sup_{w \in W^e} |h_w|(\mathcal{N}(w) + 1)^n.$$

The completion of  $\mathcal{H}$  with respect to these norms is the Schwartz algebra  $\mathcal{S} = \mathcal{S}(\mathcal{R}, q)$ . It is known from [Opd, Section 6.2] that it is a Fréchet algebra. The  $\Gamma$ -action on  $\mathcal{H}$  extends continuously to  $\mathcal{S}$ , so the crossed product algebra  $\mathcal{S}(\mathcal{R}, q) \rtimes \Gamma$  is well-defined. By [Opd, Lemma 2.20] a finite dimensional  $\mathcal{H} \rtimes \Gamma$ -representation is tempered if and only if it extends continuously to an  $\mathcal{S} \rtimes \Gamma$ -representation.

We define a  $*$ -operation and a trace on  $\mathcal{H}(\mathcal{R}, q)$  by

$$\begin{aligned} \left(\sum_{w \in W^e} c_w N_w\right)^* &= \sum_{w \in W^e} \overline{c_w} N_{w^{-1}}, \\ \tau\left(\sum_{w \in W^e} c_w N_w\right) &= c_e. \end{aligned}$$

Since  $q(s_\alpha) > 0$ ,  $*$  preserves the relations (6) and defines an anti-involution of  $\mathcal{H}(\mathcal{R}, q)$ . The set  $\{N_w : w \in W^e\}$  is an orthonormal basis of  $\mathcal{H}(\mathcal{R}, q)$  for the inner product

$$\langle h_1, h_2 \rangle = \tau(h_1^* h_2).$$

This gives  $\mathcal{H}(\mathcal{R}, q)$  the structure of a Hilbert algebra. The Hilbert space completion  $L^2(\mathcal{R})$  of  $\mathcal{H}(\mathcal{R}, q)$  is a module over  $\mathcal{H}(\mathcal{R}, q)$ , via left multiplication. Moreover every  $h \in \mathcal{H}(\mathcal{R}, q)$  acts as a bounded linear operator [Opd, Lemma 2.3]. The reduced  $C^*$ -algebra of  $\mathcal{H}(\mathcal{R}, q)$  [Opd, §2.4], denoted  $C_r^*(\mathcal{R}, q)$ , is defined as the closure of  $\mathcal{H}(\mathcal{R}, q)$  in the algebra of bounded linear operators on  $L^2(\mathcal{R})$ . By [Opd, Theorem 6.1]

$$\mathcal{H}(\mathcal{R}, q) \subset \mathcal{S}(\mathcal{R}, q) \subset C_r^*(\mathcal{R}, q).$$

As in (11), we can extend this to a  $C^*$ -algebra  $C_r^*(\mathcal{R}, q) \rtimes \Gamma$ , provided that  $q$  is  $\Gamma$ -invariant.

Let us recall some background about  $C_r^*(\mathcal{R}, q) \rtimes \Gamma$ , mainly from [Opd, Sol1]. It follows from [DeOp, Corollary 5.7] that it is a finite type I  $C^*$ -algebra and that  $\text{Irr}(C_r^*(\mathcal{R}, q))$  is precisely the tempered part of  $\text{Irr}(\mathcal{H}(\mathcal{R}, q))$ . According to [Opd, Theorem 4.23] all irreducible  $\mathcal{S}(\mathcal{R}, q) \rtimes \Gamma$ -representations extend continuously to  $C_r^*(\mathcal{R}, q) \rtimes \Gamma$ . Hence we can regard the representation theory of  $C_r^*(\mathcal{R}, q) \rtimes \Gamma$  as the tempered unitary representation theory of  $\mathcal{H}(\mathcal{R}, q) \rtimes \Gamma$ .

The structure of  $C_r^*(\mathcal{R}, q) \rtimes \Gamma$  is described in terms of parabolically induced representations. As induction data we use triples  $(Q, \delta, t)$  where  $Q \subset \Delta$ ,  $\delta \in \text{Irr}_{L^2}(\mathcal{H}_Q)$  and  $t \in T^Q$ . We regard two triples  $(Q, \delta, t)$  and  $(Q', \delta', t')$  as equivalent if  $Q = Q'$ ,  $t = t'$  and  $\delta \cong \delta'$ . Notice that  $\mathcal{H}_Q$  comes from a semisimple root datum, so it can have discrete series representations. We inflate such a representation to  $\mathcal{H}^Q$  via the projection (7). To a triple  $(Q, \delta, t)$  we associate the  $\mathcal{H} \rtimes \Gamma$ -representation

$$\pi^\Gamma(Q, \delta, t) = \text{ind}_{\mathcal{H}^Q}^{\mathcal{H} \rtimes \Gamma}(\delta \circ \psi_t).$$

(When  $\Gamma = 1$ , we often suppress it from these and similar notations.) For  $t \in T_{\text{un}}^Q = T^Q \cap T_{\text{un}}$  these representations extend continuously to the respective  $C^*$ -completions of the involved algebras. Let  $\Xi_{\text{un}}$  be the set of triples  $(Q, \delta, t)$  as above, such that moreover  $t \in T_{\text{un}}$ . Considering  $Q$  and  $\delta$  as discrete variables, we regard  $\Xi_{\text{un}}$  as a disjoint union of finitely many compact real tori (of different dimensions).



Let  $\mathcal{V}_{\Xi}^{\Gamma}$  be the vector bundle over  $\Xi_{\text{un}}$ , whose fibre at  $\xi = (Q, \delta, t)$  is the vector space underlying  $\pi^{\Gamma}(Q, \delta, t)$ . That vector space is independent of  $t$ , so the vector bundle is trivial. Let  $\text{End}(\mathcal{V}_{\Xi}^{\Gamma})$  be the algebra bundle with fibres  $\text{End}_{\mathbb{C}}(\pi^{\Gamma}(Q, \delta, t))$ . These data give rise to a canonical map

$$(12) \quad \begin{array}{ccc} \mathcal{H}(\mathcal{R}, q) \rtimes \Gamma & \rightarrow & \mathcal{O}(\Xi; \text{End}(\mathcal{V}_{\Xi}^{\Gamma})) \\ h & \mapsto & (\xi \mapsto \pi^{\Gamma}(\xi)(h)) \end{array}$$

which we refer to as the Fourier transform. By [Opd, Lemma 2.22] every discrete series representation is unitary, so  $V_{\delta}$  carries an  $\mathcal{H}_Q$ -invariant inner product and  $\text{End}_{\mathbb{C}}(V_{\delta})$  has a natural  $*$ -operation. For any  $t \in T^Q$  this becomes an  $\mathcal{H}^Q$ -invariant nondegenerate pairing between  $\delta \circ \phi_t$  and  $\delta \circ \phi_{t|t|^{-2}}$ . By [Opd, Proposition 4.19] this extends canonically to an inner product on the vector space

$$(13) \quad \pi^{\Gamma}(Q, \delta, t) = \Gamma \rtimes \mathcal{H}(W, q) \otimes_{\mathcal{H}(W_Q, q)} V_{\delta}.$$

That yields an involution on  $\text{End}_{\mathbb{C}}(\pi^{\Gamma}(Q, \delta, t))$  and a nondegenerate  $\mathcal{H} \rtimes \Gamma$ -invariant pairing between  $\pi^{\Gamma}(Q, \delta, t)$  and  $\pi^{\Gamma}(Q, \delta, t|t|^{-2})$ .

The algebra  $\mathcal{O}(\Xi; \text{End}(\mathcal{V}_{\Xi}^{\Gamma}))$  is endowed with the involution

$$(14) \quad (f^*)(Q, \delta, t) = f(Q, \delta, t|t|^{-2})^*.$$

With respect to this involution, (12) is a  $*$ -homomorphism.

There exists a finite groupoid  $\mathcal{G}$  which acts on  $\text{End}(\mathcal{V}_{\Xi}^{\Gamma})$ . It is made from elements of  $W \rtimes \Gamma$  and of  $K_Q := T_Q \cap T^Q$ . More precisely, its base space is the power set of  $\Delta$ , and for  $Q, Q' \subseteq \Delta$  the collection of arrows from  $Q$  to  $Q'$  is

$$(15) \quad \mathcal{G}_{QQ'} = \{(g, u) : g \in \Gamma \rtimes W, u \in K_Q, g(Q) = Q'\}.$$

Whenever it is defined, the multiplication in  $\mathcal{G}$  is

$$(g', u') \cdot (g, u) = (g'g, g^{-1}(u')u).$$

In particular, writing  $W\Gamma(Q, Q) = \{w \in W\Gamma : w(Q) = Q\}$ , we have the group

$$(16) \quad \mathcal{G}_{QQ} = W\Gamma(Q, Q) \rtimes K_Q.$$

Usually we will write elements of  $\mathcal{G}$  simply as  $gu$ . For  $\gamma \in \Gamma W$  with  $\gamma(Q) = Q' \subset \Delta$  there are algebra isomorphisms

$$(17) \quad \begin{array}{lll} \psi_{\gamma} : \mathcal{H}_Q \rightarrow \mathcal{H}_{Q'}, & \theta_{x_Q} N_w & \mapsto \theta_{\gamma(x_Q)} N_{\gamma w \gamma^{-1}}, \\ \psi_{\gamma} : \mathcal{H}^Q \rightarrow \mathcal{H}^{Q'}, & \theta_x N_w & \mapsto \theta_{\gamma x} N_{\gamma w \gamma^{-1}}. \end{array}$$

The groupoid  $\mathcal{G}$  acts from the left on  $\Xi_{\text{un}}$  by

$$(18) \quad (g, u) \cdot (Q, \delta, t) := (g(Q), \delta \circ \psi_u^{-1} \circ \psi_g^{-1}, g(ut)),$$

the action being defined if and only if  $g(Q) \subset \Delta$ . Suppose that  $g(Q) = Q' \subset \Delta$  and  $\delta' \cong \delta \circ \psi_u^{-1} \circ \psi_g^{-1}$ . By [Opd, Theorem 4.33] and [Sol1, Theorem 3.1.5] there exists an intertwining operator

$$(19) \quad \pi^{\Gamma}(gu, Q, \delta, t) \in \text{Hom}_{\mathcal{H}(\mathcal{R}, q) \rtimes \Gamma}(\pi^{\Gamma}(Q, \delta, t), \pi^{\Gamma}(Q', \delta', g(ut))),$$

which depends algebraically on  $t \in T_{\text{un}}^Q$ . Then the action of  $\mathcal{G}$  on the continuous sections  $C(\Xi_{\text{un}}; \text{End}(\mathcal{V}_{\Xi}^{\Gamma}))$  is given by

$$(20) \quad (g \cdot f)(g\xi) = \pi^{\Gamma}(g, \xi) f(\xi) \pi^{\Gamma}(g, \xi)^{-1} \quad g \in \mathcal{G}_{QQ}, \xi = (Q, \delta, t).$$

The next result is the Plancherel isomorphism for affine Hecke algebras, proven in [DeOp, Theorem 5.3 and Corollary 5.7] and [Sol1, Theorem 3.2.2].

**Theorem 1.1.** *The map Fourier transform (12) induces  $*$ -homomorphisms*

$$\begin{aligned}\mathcal{H}(\mathcal{R}, q) \rtimes \Gamma &\rightarrow \mathcal{O}(\Xi; \text{End}(\mathcal{V}_{\Xi}^{\Gamma}))^{\mathcal{G}}, \\ \mathcal{S}(\mathcal{R}, q) \rtimes \Gamma &\rightarrow C^{\infty}(\Xi_{\text{un}}; \text{End}(\mathcal{V}_{\Xi}^{\Gamma}))^{\mathcal{G}}, \\ C_r^*(\mathcal{R}, q) \rtimes \Gamma &\rightarrow C(\Xi_{\text{un}}; \text{End}(\mathcal{V}_{\Xi}^{\Gamma}))^{\mathcal{G}}.\end{aligned}$$

*The first is injective, the second is an isomorphism of Fréchet algebras and the third is an isomorphism of  $C^*$ -algebras.*

For  $q = 1$  this simplifies to the well-known isomorphisms

$$\begin{aligned}(21) \quad \mathcal{H}(\mathcal{R}, 1) \rtimes \Gamma &= \mathcal{O}(T) \rtimes W\Gamma \rightarrow \mathcal{O}(T; \text{End}_{\mathbb{C}}(\mathbb{C}[W\Gamma]))^{W\Gamma}, \\ \mathcal{S}(\mathcal{R}, 1) \rtimes \Gamma &= C^{\infty}(T_{\text{un}}) \rtimes W\Gamma \rightarrow C^{\infty}(T_{\text{un}}; \text{End}_{\mathbb{C}}(\mathbb{C}[W\Gamma]))^{W\Gamma}, \\ C_r^*(\mathcal{R}, 1) \rtimes \Gamma &= C(T_{\text{un}}) \rtimes W\Gamma \rightarrow C(T_{\text{un}}; \text{End}_{\mathbb{C}}(\mathbb{C}[W\Gamma]))^{W\Gamma}.\end{aligned}$$

Unfortunately, the bookkeeping in Theorem 1.1 is not entirely suitable for our purposes. Namely, in some case that we will encounter, the appropriate parabolic subalgebras of  $\mathcal{H}(\mathcal{R}, q) \rtimes \Gamma$  are not  $\mathcal{H}(\mathcal{R}^Q, q^Q)$ , but  $\mathcal{H}(\mathcal{R}^Q, q^Q) \rtimes \Gamma_Q$  for some subgroup  $\Gamma_Q \subset \Gamma$ . In those cases we should rather use induction data based on  $\text{Irr}_{L^2}(\mathcal{H}^Q \rtimes \Gamma_Q)$  than based on  $\text{Irr}_{L^2}(\mathcal{H}_Q)$  or  $\text{Irr}_{L^2}(\mathcal{H}^Q)$ . Here we allow  $\Gamma_{\Delta} \subsetneq \Gamma$ .

To make this work, we need some assumptions on the groups  $\Gamma_Q$  for  $Q \subset \Delta$ .

**Condition 1.2.** (1)  $\Gamma_Q \subset \Gamma_{Q'}$  if  $Q \subset Q'$ ;  
(2) the action of  $\Gamma_Q$  on  $T$  stabilizes  $T_Q$  and  $T^Q$ ;  
(3)  $\Gamma_Q$  acts on  $T^Q$  by multiplication with elements of  $K_Q$ .

Notice that these conditions entail that  $\Gamma_{\emptyset}$  acts trivially on  $\mathcal{O}(T) = \mathcal{H}^{\emptyset}$ , so

$$(22) \quad \text{Irr}(\mathcal{H}^{\emptyset} \rtimes \Gamma_{\emptyset}) \cong T \times \text{Irr}(\Gamma_{\emptyset}).$$

**Remark 1.3.** Often there is a larger root system  $\tilde{R} \supset R$  in  $X$ , such that  $W_Q \Gamma_Q$  is contained in the parabolic subgroup of  $W(\tilde{R})$  associated to  $\tilde{R} \cap \mathbb{Q}Q$ . Then parts (2) and (3) of Condition 1.2 are automatically satisfied (and part (1) is usually obvious).

Under Conditions 1.2  $\Gamma_Q$  commutes with  $K_Q$ , so the groupoid  $\mathcal{G}^Q$  for  $\mathcal{H}^Q \rtimes \Gamma_Q$  satisfies  $\mathcal{G}_{Q_Q}^Q = \Gamma_Q \times K_Q$ . The conditions also entail that the projection  $\mathcal{H}^Q \rightarrow \mathcal{H}_Q$  and the isomorphisms  $\phi_t : \mathcal{H}^Q \rightarrow \mathcal{H}^Q$  ( $t \in T^Q$ ) are  $\Gamma_Q$ -equivariant, so they extend to algebra homomorphisms

$$(23) \quad \mathcal{H}^Q \rtimes \Gamma_Q \rightarrow \mathcal{H}_Q \rtimes \Gamma_Q \quad \text{and} \quad \phi_t : \mathcal{H}^Q \rtimes \Gamma_Q \rightarrow \mathcal{H}^Q \rtimes \Gamma_Q \quad (t \in T^Q).$$

Via the first map of (23) we can inflate any representation of  $\mathcal{H}_Q \rtimes \Gamma_Q$  to  $\mathcal{H}^Q \rtimes \Gamma_Q$ , which we often do tacitly. For any representation  $\pi$  of  $\mathcal{H}^Q \rtimes \Gamma_Q$  and any  $t \in T^Q$  we write

$$\pi \otimes t = \pi \circ \phi_t \in \text{Mod}(\mathcal{H}^Q \rtimes \Gamma_Q).$$

**Lemma 1.4.** (a) *Every irreducible  $\mathcal{H}^Q \rtimes \Gamma_Q$ -representation is of the form  $\pi_Q \otimes t^Q$  for some  $\pi_Q \in \text{Irr}(\mathcal{H}_Q \rtimes \Gamma_Q)$  and  $t^Q \in T^Q$ .*  
(b)  *$\pi_Q \otimes t^Q$  is tempered if and only if  $\pi$  is tempered and  $t^Q \in T_{\text{un}}^Q$ .*  
(c)  *$\pi_Q \otimes t^Q$  is essentially discrete series if and only if  $\pi_Q$  is discrete series.*

*Proof.* (a) First we consider the situation without  $\Gamma_Q$ . Let  $\pi \in \text{Irr}(\mathcal{H}^Q)$  with central character  $W_Q t \in T/W_Q$ . The group  $W_Q = W(R_Q)$  acts trivially on  $T^Q$ , so  $W_Q t =$

$t^Q W_Q t_Q$  for some  $t^Q \in T^Q, t_Q \in T_Q$ . Then  $\pi \otimes (t^Q)^{-1}$  factors through  $\mathcal{H}^Q \rightarrow \mathcal{H}_Q$  (say as  $\pi_Q$ ), and  $\pi = \pi_Q \otimes t^Q$ .

To include  $\Gamma_Q$  we use Clifford theory [RaRa, Theorem A.6]. It says that every irreducible  $\mathcal{H}^Q \rtimes \Gamma_Q$ -representation is of the form

$$\pi \rtimes \sigma := \text{ind}_{\mathcal{H}^Q \rtimes \Gamma_{Q,\pi}}^{\mathcal{H}^Q \rtimes \Gamma_Q} (\pi \otimes \sigma).$$

Here  $\Gamma_{Q,\pi}$  is the stabilizer of  $\pi \in \text{Irr}(\mathcal{H}^Q)$  in  $\Gamma_Q$  and  $(\sigma, V_\sigma)$  is an irreducible representation of a twisted group algebra of  $\Gamma_{Q,\pi}$ . If  $\mathcal{O}(T)$  acts by  $t^Q t_1$  on a vector subspace  $V_1 \subset V_\pi$ , then for  $\gamma \in \Gamma_Q$  it acts by the character  $\gamma^{-1}(t^Q t_1)$  on  $N_\gamma(V_1 \otimes V_\sigma)$ . By Condition 1.2.(3)  $\gamma^{-1}(t^Q t_1) = t^Q \gamma^{-1}(t_1)$ . Hence  $(\pi \rtimes \sigma) \otimes (t^Q)^{-1}$  factors through  $\mathcal{H}^Q \rtimes \Gamma_Q \rightarrow \mathcal{H}_Q \rtimes \Gamma_Q$  as  $\pi_Q \rtimes \sigma$ , and

$$\pi \rtimes \sigma = (\pi_Q \rtimes \sigma) \otimes t^Q.$$

(b) As  $T = T^Q T_Q$  with  $T^Q \cap T_Q \subset T_{\text{un}}$ ,  $T_{rs} = T_{rs}^Q \times T_{Q,rs}$  and (with respect to  $\mathcal{R}^Q$ )  $T_{rs}^- = \{1\} \times T_{Q,rs}^-$ . Also  $|\text{Wt}(\pi \otimes t^Q)| = |t^Q| |\text{Wt}(\pi)|$ . These imply the result.

(c) This is obvious from  $\text{Wt}(\pi \otimes t^Q) = t^Q \text{Wt}(\pi)$ .  $\square$

With Lemma 1.4 in mind we define new induction data. They are triples  $(Q, \sigma, t)$  with  $Q \subset \Delta$ ,  $t \in T^Q$  and  $\sigma \in \text{Irr}_{L^2}(\mathcal{H}_Q \rtimes \Gamma_Q)$ . To such a triple we associate the representation

$$\pi(Q, \sigma, t) = \text{ind}_{\mathcal{H}^Q \rtimes \Gamma_Q}^{\mathcal{H} \rtimes \Gamma} (\sigma \otimes t).$$

The underlying vector space does not depend on  $t$ , we denote it by  $V_{Q,\sigma}$ . There is a natural homomorphism

$$(24) \quad \begin{array}{ccc} \mathcal{H}(\mathcal{R}, q) \rtimes \Gamma & \rightarrow & \mathcal{O}(T^Q) \otimes \text{End}_{\mathbb{C}}(V_{Q,\sigma}) \\ h & \mapsto & (t \mapsto \pi(Q, \sigma, t)(h)). \end{array}$$

We refer to the system of these maps, for all  $Q$  and  $\sigma$ , as the Fourier transform for  $\mathcal{H}(\mathcal{R}, q) \rtimes \Gamma$ .

We keep the same groupoid  $\mathcal{G}$  as before, it also acts on the new triples via (18). The recipe for the intertwining operators from [Opd, §4] and [Sol1, Theorem 3.1.5] remains valid so we get

$$\pi(gu, Q, \sigma, t) \in \text{Hom}_{\mathcal{H} \rtimes \Gamma} (\pi(Q, \sigma, t), \pi(Q', \sigma', g(ut)))$$

with the same properties as in (19). With these notions we can formulate our variation on the Plancherel isomorphism (Theorem 1.1).

**Proposition 1.5.** *The Fourier transform from (24) induces isomorphisms of Fréchet \*-algebras*

$$\begin{array}{ccc} \mathcal{S}(\mathcal{R}, q) \rtimes \Gamma & \rightarrow & \left( \bigoplus_{Q,\sigma} C^\infty(T_{\text{un}}^Q; \text{End}_{\mathbb{C}}(V_{Q,\sigma})) \right)^{\mathcal{G}}, \\ C_r^*(\mathcal{R}, q) \rtimes \Gamma & \rightarrow & \left( \bigoplus_{Q,\sigma} C(T_{\text{un}}^Q; \text{End}_{\mathbb{C}}(V_{Q,\sigma})) \right)^{\mathcal{G}}. \end{array}$$

*Proof.* We will analyse the right hand side of Theorem 1.1 (for the Schwartz algebras) and compare it with the current setting.

First we consider essentially discrete series representations of  $\mathcal{H}^Q \rtimes \Gamma_Q$ . Pick  $\delta_1 \in \text{Irr}_{L^2}(\mathcal{H}_Q)$  and  $t_1 \in T_{\text{un}}^Q$ . We note that  $\text{ind}_{\mathcal{H}^Q}^{\mathcal{H}^Q \rtimes \Gamma_Q} (\delta \otimes t_1)$  is unitary and essentially

discrete series, because  $\Gamma_Q$  stabilizes  $Q$ . For  $\sigma \in \text{Irr}_{L^2}(\mathcal{H}^Q \rtimes \Gamma_Q)$  we write  $(\sigma, t) > (\delta_1, t_1)$  if

$$\text{Hom}_{\mathcal{H}^Q \rtimes \Gamma_Q}(\sigma \otimes t, \text{ind}_{\mathcal{H}^Q}^{\mathcal{H}^Q \rtimes \Gamma_Q}(\delta \otimes t_1)) \cong \text{Hom}_{\mathcal{H}^Q}(\sigma \otimes t, \delta \otimes t_1)$$

is nonzero. Since  $\Gamma_Q$  is finite, the set of such  $(\sigma, t)$  is finite. Hence the map

$$\bigoplus_{(\sigma, t) > (\delta_1, t_1)} \sigma \circ \psi_t : \mathcal{H}^Q \rtimes \Gamma_Q \rightarrow \bigoplus_{(\sigma, t) > (\delta_1, t_1)} \text{End}_{\mathbb{C}}(V_\sigma)$$

is surjective. Write  $\mathcal{G}_{QQ}^Q \delta_1 = \{\delta_i\}_i$ . The summand of  $C^\infty(\Xi_{Q, \text{un}}; \text{End}(\mathcal{V}_{\Xi_Q}^{\Gamma_Q}))^{\mathcal{G}^Q}$  associated to  $(Q, \delta_1)$  is

$$(25) \quad \left( \bigoplus_i C^\infty(T_{\text{un}}^Q; \text{End}_{\mathbb{C}}(\pi^{\Gamma_Q}(Q, \delta_i, t_i))) \right)^{\mathcal{G}_{QQ}^Q}.$$

The specialization of (25) at  $\mathcal{G}_{QQ}^Q(Q, \delta_1, t_1)$  is also  $\bigoplus_{(\sigma, t) > (\delta_1, t_1)} \text{End}_{\mathbb{C}}(V_\sigma)$ , for that specialization is just

$$\text{ind}_{\mathcal{H}^Q}^{\mathcal{H}^Q \rtimes \Gamma_Q}(\delta \circ \psi_{t_1})(\mathcal{S}(\mathcal{R}^Q, q^Q) \rtimes \Gamma_Q).$$

Let  $\{\sigma_j\}_j$  be the members of  $\text{Irr}_{L^2}(\mathcal{H}_Q \rtimes \Gamma_Q)$  contained in  $\text{ind}_{\mathcal{H}^Q}^{\mathcal{H}_Q \rtimes \Gamma_Q}(\delta_1 \circ \psi_u)$  for some  $u \in K_Q$ . This set is stable under  $\mathcal{G}_{QQ}^Q = \Gamma_Q \rtimes K_Q$ . The summand of  $\left( \bigoplus_\sigma C^\infty(T_{\text{un}}^Q; \text{End}_{\mathbb{C}}(V_\sigma)) \right)^{\mathcal{G}^Q}$  corresponding to the  $\sigma_j$  is

$$(26) \quad \left( \bigoplus_j C^\infty(T_{\text{un}}^Q; \text{End}_{\mathbb{C}}(V_{\sigma_j})) \right)^{\mathcal{G}_{QQ}^Q}.$$

Specializing this algebra at all  $(\sigma, t) > (\delta_1, t_1)$  gives a surjection from (26) to  $\bigoplus_{(\sigma, t) > (\delta_1, t_1)} \text{End}_{\mathbb{C}}(V_\sigma)$ .

Let us compare (26) with (25). Both are algebras of smooth section of (trivial) algebra bundles, and specialization at the points associated to  $(\delta_1, t_1)$  yields the same algebra in both cases. This holds for any  $t_1 \in T_{\text{un}}^Q$  and that accounts for all base points of these algebra bundles, so (25) and (26) are isomorphic. Moreover the isomorphism is canonical: it is the composition of the inverse of the map in Theorem 1.1 and the map induced by (24) (both for  $\mathcal{H}^Q \rtimes \Gamma_Q$ ).

Thus, for every  $Q \subset \Delta$  we get a canonical isomorphism

$$(27) \quad \left( \bigoplus_{\sigma \in \text{Irr}_{L^2}(\mathcal{H}_Q \rtimes \Gamma_Q)} C^\infty(T_{\text{un}}^Q; \text{End}_{\mathbb{C}}(V_\sigma)) \right)^{\mathcal{G}_{QQ}^Q} \cong \left( \bigoplus_{\delta \in \text{Irr}_{L^2}(\mathcal{H}_Q)} C^\infty(T_{\text{un}}^Q; \text{End}_{\mathbb{C}}(\mathbb{C}[\Gamma_Q] \otimes V_\delta)) \right)^{\mathcal{G}_{QQ}^Q}.$$

To obtain the right hand side of Theorem 1.1 from (27), we must apply  $\text{ind}_{\mathcal{H}^Q \rtimes \Gamma_Q}^{\mathcal{H} \rtimes \Gamma}$  to  $\mathbb{C}[\Gamma_Q] \otimes V_\delta \cong \text{ind}_{\mathcal{H}^Q}^{\mathcal{H}^Q \rtimes \Gamma_Q}(\delta \otimes t)$  and then take invariants with respect to the larger groupoid  $\mathcal{G} \supset \mathcal{G}^Q$ . The formula [Sol1, (3.12)] is the same for  $\mathcal{G}^Q$  and for  $\mathcal{G}$ , so the intertwiners associated to elements of  $\mathcal{G}^Q$  need not be adjusted in this process.

With exactly the same procedure we can turn (27) into the right hand side of the current proposition. The intertwining operators associated to elements of  $\mathcal{G}$  agree

under the isomorphisms obtained from (27) by applying  $\text{ind}_{\mathcal{H}^Q \rtimes \Gamma_Q}^{\mathcal{H} \rtimes \Gamma}$ , because in both settings they were constructed with [Sol1, (3.12) and Theorem 3.1.5]. Consequently

$$\left( \bigoplus_{Q, \sigma} C^\infty(T_{\text{un}}^Q; \text{End}_{\mathbb{C}}(V_{Q, \sigma})) \right)^{\mathcal{G}} \cong C^\infty(\Xi_{\text{un}}; \text{End}(\mathcal{V}_{\Xi}^\Gamma))^{\mathcal{G}},$$

proving the proposition for the Schwartz algebras. For  $C_r^*(\mathcal{R}, q) \rtimes \Gamma$  one can use the same argument, with everywhere  $C^\infty$  replaced by continuous functions.  $\square$

Choose representatives  $Q$  for  $\mathcal{P}(\Delta)$  modulo  $W\Gamma$ -association. For every such  $Q$  we choose representatives  $\sigma$  for the action of  $\mathcal{G}_{QQ}$  on  $\text{Irr}_{L^2}(\mathcal{H}_Q \rtimes \Gamma_Q)$ . By Lemma 1.4 these  $\sigma$  also form representatives for the action of  $\mathcal{G}_{QQ} \times T_{\text{un}}^Q$  on  $\text{Irr}_{L^2}(\mathcal{H}^Q \rtimes \Gamma_Q)$ . We denote the resulting set of representatives of pairs by  $(Q, \sigma)/\mathcal{G}$ . Let  $\mathcal{G}_{Q, \sigma}$  be the setwise stabilizer of  $(Q, \sigma, T_{\text{un}}^Q)$  in the group  $\mathcal{G}_{QQ}$ . Proposition 1.5 can be rephrased as isomorphisms

$$(28) \quad \begin{aligned} \mathcal{S}(\mathcal{R}, q) \rtimes \Gamma &\rightarrow \bigoplus_{(Q, \sigma)/\mathcal{G}} C^\infty(T_{\text{un}}^Q; \text{End}_{\mathbb{C}}(V_{Q, \sigma}))^{\mathcal{G}_{Q, \sigma}}, \\ C_r^*(\mathcal{R}, q) \rtimes \Gamma &\rightarrow \bigoplus_{(Q, \sigma)/\mathcal{G}} C(T_{\text{un}}^Q; \text{End}_{\mathbb{C}}(V_{Q, \sigma}))^{\mathcal{G}_{Q, \sigma}}. \end{aligned}$$

Sometimes we have to consider the opposite algebra  $(\mathcal{H}(\mathcal{R}, q) \rtimes \Gamma)^{\text{op}}$  and its completions. It is, morally, clear that all the previous results can also be developed for right  $\mathcal{H} \rtimes \Gamma$ -modules, that is, for  $(\mathcal{H} \rtimes \Gamma)^{\text{op}}$ -modules. However, none of that has been written down, so we prefer more steady ground.

For every  $\mathcal{H} \rtimes \Gamma$ -representation  $(\pi, V_\pi)$ , the full linear dual  $V_\pi^*$  becomes a  $(\mathcal{H} \rtimes \Gamma)^{\text{op}}$ -representation  $\pi^*$  by

$$\pi^*(h^{\text{op}})\lambda = \lambda \circ \pi(h).$$

This sets up a bijection between finite dimensional left and right modules of  $\mathcal{H} \rtimes \Gamma$ . In view of the canonical inner products from on the spaces (13), this bijection commutes with induction from parabolic subalgebras.

For infinite dimensional representations there is often some choice for which dual space of  $V_\pi$  we use here. In particular, when  $V_\pi$  is a Hilbert space we can use  $V_\pi$  also as dual space. With this convention one checks easily that  $\pi$  is unitary if and only if  $\pi^*$  is unitary.

The  $\mathcal{O}(T)$ -weights of  $\pi^*$  are the same as for  $\pi$ , so  $\pi^*$  is tempered or (essentially) discrete series if and only if  $\pi$  is so. Thus the pairs  $(Q, \sigma)$  with  $\sigma \in \text{Irr}_{L^2}(\mathcal{H}_Q \rtimes \Gamma_Q)$  are in natural bijection with the pairs  $(Q, \sigma^*)$  in

$$(29) \quad \bigcup_{Q \subset \Delta} \text{Irr}_{L^2}((\mathcal{H}(\mathcal{R}_Q, q_Q) \rtimes \Gamma_Q)^{\text{op}}).$$

The bijection is  $\mathcal{G}$ -equivariant for the  $\mathcal{G}$ -action on (29) as in (18). Hence  $\mathcal{G}_{Q, \sigma} = \mathcal{G}_{Q, \sigma^*}$  and we can take  $(Q, \sigma^*)/\mathcal{G}$  to be the image of  $(Q, \sigma)/\mathcal{G}$ .

**Lemma 1.6.** *The Fourier transform for right  $\mathcal{H}(\mathcal{R}, q) \rtimes \Gamma$ -modules induces isomorphisms of Fréchet \*-algebras*

$$\begin{aligned} (\mathcal{S}(\mathcal{R}, q) \rtimes \Gamma)^{\text{op}} &\rightarrow \bigoplus_{(Q, \sigma^*)/\mathcal{G}} C^\infty(T_{\text{un}}^Q; \text{End}_{\mathbb{C}}(V_{Q, \sigma^*}))^{\mathcal{G}_{Q, \sigma^*}}, \\ (C_r^*(\mathcal{R}, q) \rtimes \Gamma)^{\text{op}} &\rightarrow \bigoplus_{(Q, \sigma^*)/\mathcal{G}} C(T_{\text{un}}^Q; \text{End}_{\mathbb{C}}(V_{Q, \sigma^*}))^{\mathcal{G}_{Q, \sigma^*}}. \end{aligned}$$

*Proof.* The opposite algebra of

$$\text{End}_{\mathbb{C}}(V_{Q, \sigma}) = \text{End}_{\mathbb{C}}(\text{ind}_{\mathcal{H}^Q \rtimes \Gamma_Q}^{\mathcal{H} \rtimes \Gamma} V_\delta)$$

is naturally isomorphic to  $\text{End}_{\mathbb{C}}(V_{Q,\sigma}^*)$ , which by [Opd, Proposition 4.19] is canonically isomorphic with

$$\text{End}_{\mathbb{C}}(\text{ind}_{\mathcal{H}^Q \rtimes \Gamma_Q}^{\mathcal{H} \rtimes \Gamma}(V_{\sigma}^*)) = \text{End}_{\mathbb{C}}(V_{Q,\sigma^*}).$$

For  $g \in \mathcal{G}_{Q,\sigma}$  we take  $\pi(g, Q, \sigma^*, t|t|^{-2})$  to be the transpose inverse of  $\pi(g, Q, \sigma, t)$ . Thus an element of  $C(T_{\text{un}}^Q; \text{End}_{\mathbb{C}}(V_{Q,\sigma}))$  is  $\mathcal{G}_{Q,\sigma}$ -invariant if and only if its transpose in  $C(T_{\text{un}}^Q; \text{End}_{\mathbb{C}}(V_{Q,\sigma^*}))$  is  $\mathcal{G}_{Q,\sigma^*}$ -invariant for the action

$$(g \cdot f)(g(Q, \sigma^*, t)) = \pi(g, Q, \sigma^*, t)f(Q, \sigma^*, t)\pi(g, Q, \sigma^*, t)^{-1}.$$

Now we take the opposite algebras in Theorem 1.1 and we find the desired isomorphisms.

The implementing algebra homomorphisms are given by transpose, the Fourier transform from (24) and again transpose, which works out to the Fourier transform for  $(\mathcal{H}(\mathcal{R}, q) \rtimes \Gamma)^{\text{op}}$ -modules. Since the correspondence between left and right  $\mathcal{H}(\mathcal{R}, q) \rtimes \Gamma$ -modules preserves unitarity, the latter Fourier transform is still a  $*$ -homomorphism.  $\square$

## 2. GROUP ALGEBRAS OF REDUCTIVE $p$ -ADIC GROUPS

Let  $F$  be a local non-archimedean field and let  $G = \mathcal{G}(F)$  be a connected reductive algebraic group over  $F$ . We endow  $G$  with the topology coming from the metric on  $F$  and we fix a Haar measure on  $G$ . Let  $\mathcal{H}(G)$  be the Hecke algebra of  $G$ , the convolution algebra of locally constant compactly supported functions  $G \rightarrow \mathbb{C}$ . The product on  $\mathcal{H}(G)$  is convolution (with respect to the Haar measure). Let  $\mathcal{S}(G)$  be the Harish-Chandra–Schwartz algebra of  $G$ , as defined in [HC] and [Wal, §III.6]. Let  $C_r^*(G)$  be the reduced  $C^*$ -algebra of  $G$ , the completion of  $\mathcal{H}(G)$  in the algebras of bounded linear operators on the Hilbert space  $L^2(G)$ . By [Vig, Theorem 29] there are dense inclusions

$$\mathcal{H}(G) \subset \mathcal{S}(G) \subset C_r^*(G).$$

The  $*$ -operation on these algebras is just  $f^*(g) = \overline{f(g^{-1})}$ .

Let  $L = \mathcal{L}(F)$  be a Levi subgroup  $G$  and let  $(\sigma, V_{\sigma}) \in \text{Irr}(L)$  be an irreducible tempered supercuspidal  $L$ -representation. Let  $X_{\text{nr}}(L)$  be the group of unramified characters  $L \rightarrow \mathbb{C}^{\times}$  and let  $X_{\text{unr}}(L)$  be the subgroup of unitary unramified characters. Recall that the inertial equivalence class of the pair  $(L, \sigma)$  consists of all pairs of the form

$$(gLg^{-1}, (g \cdot \sigma) \otimes \chi), \text{ where } g \in G \text{ and } \chi \in X_{\text{nr}}(gLg^{-1}).$$

We write  $\mathfrak{s} = [L, \sigma]_G$  and call this an inertial equivalence class for  $G$ . It gives rise to a subset  $\text{Irr}(G)^{\mathfrak{s}} \subset \text{Irr}(G)$ , namely all those irreducible smooth  $G$ -representations whose supercuspidal support lies in  $\mathfrak{s}$ . This in turn is used to define a subcategory  $\text{Rep}(G)^{\mathfrak{s}}$  of  $\text{Rep}(G)$ , namely those smooth  $G$ -representations all whose irreducible constituents lie in  $\text{Irr}(G)^{\mathfrak{s}}$ .

Let  $\mathfrak{B}(G)$  be the set of all such inertial equivalence classes  $\mathfrak{s}$ . By [BeDe, Corollaire 3.9] it is countably infinite, unless  $G = 1$ . The Bernstein decomposition [BeDe, Theorem 2.10] says that

$$(30) \quad \begin{aligned} \text{Rep}(G) &= \prod_{\mathfrak{s} \in \mathfrak{B}(G)} \text{Rep}(G)^{\mathfrak{s}}, \\ \mathcal{H}(G) &= \bigoplus_{\mathfrak{s} \in \mathfrak{B}(G)} \mathcal{H}(G)^{\mathfrak{s}}, \end{aligned}$$

where  $\mathcal{H}(G)^\mathfrak{s}$  is the two-sided ideal of  $\mathcal{H}(G)$  for which  $\text{Mod}(\mathcal{H}(G)^\mathfrak{s})$  is naturally equivalent with  $\text{Rep}(G)^\mathfrak{s}$ .

Let  $\mathcal{S}(G)^\mathfrak{s}$  (resp.  $C_r^*(G)^\mathfrak{s}$ ) be the two-sided ideal of  $\mathcal{S}(G)$  (resp.  $C_r^*(G)$ ) generated by  $\mathcal{H}(G)^\mathfrak{s}$ . Upon completion, (30) yields further Bernstein decompositions

$$(31) \quad \begin{aligned} \mathcal{S}(G) &= \bigoplus_{\mathfrak{s} \in \mathfrak{B}(G)} \mathcal{S}(G)^\mathfrak{s}, \\ C_r^*(G) &= \bigoplus_{\mathfrak{s} \in \mathfrak{B}(G)} C_r^*(G)^\mathfrak{s}. \end{aligned}$$

The latter must be interpreted as a direct sum in the Banach algebra sense: it is the completion of the algebraic direct sum with respect to the operator norm of  $C_r^*(G)$ .

For a compact open subgroup  $K$  of  $G$  we let  $\langle K \rangle$  be the corresponding idempotent of  $\mathcal{H}(G)$ . Then

$$(32) \quad \mathcal{H}(G, K) := \langle K \rangle \mathcal{H}(G) \langle K \rangle$$

is the subalgebra of  $K$ -biinvariant functions in  $\mathcal{H}(G)$ . We define  $\mathcal{S}(G, K)$  and  $C_r^*(G, K)$  analogously. For every compact open subgroup  $K$  of  $G$ ,  $\mathcal{S}(G, K)$  is a Fréchet algebra [Vig, Theorem 29]. The Schwartz algebra  $\mathcal{S}(G)$  is their union, so it is an inductive limit of Fréchet algebras.

We will focus on one Bernstein block  $\text{Rep}(G)^\mathfrak{s}$  of  $\text{Rep}(G)$ . By [BeDe, Corollaire 3.9] there exists a compact open subgroup  $K_\mathfrak{s}$  of  $G$  such that every representation in  $\text{Rep}(G)^\mathfrak{s}$  is generated by its  $K_\mathfrak{s}$ -fixed vectors. This leads to Morita equivalences

$$(33) \quad \begin{aligned} \mathcal{H}(G)^\mathfrak{s} &\sim_M \mathcal{H}(G, K_\mathfrak{s})^\mathfrak{s} &:= \mathcal{H}(G)^\mathfrak{s} \cap \mathcal{H}(G, K_\mathfrak{s}) \\ \mathcal{S}(G)^\mathfrak{s} &\sim_M \mathcal{S}(G, K_\mathfrak{s})^\mathfrak{s} &:= \mathcal{S}(G)^\mathfrak{s} \cap \mathcal{S}(G, K_\mathfrak{s}) \\ C_r^*(G)^\mathfrak{s} &\sim_M C_r^*(G, K_\mathfrak{s})^\mathfrak{s} &:= C_r^*(G)^\mathfrak{s} \cap C_r^*(G, K_\mathfrak{s}). \end{aligned}$$

We will describe the structure of  $\mathcal{S}(G)^\mathfrak{s}$  and  $\mathcal{S}(G, K_\mathfrak{s})^\mathfrak{s}$  in more detail. Let  $[L, \sigma]_L = T_\mathfrak{s} \subset \text{Irr}(L)$  be the set of  $L$ -representations of the form  $\sigma \otimes \chi$  with  $\chi \in X_{\text{nr}}(L)$ . Thus there is a finite covering of complex varieties

$$(34) \quad X_{\text{nr}}(L) \rightarrow T_\mathfrak{s} : \chi \mapsto \sigma \otimes \chi.$$

Let  $T_{\mathfrak{s}, \text{un}}$  be the subset of unitary representations in  $T_\mathfrak{s}$ , it is covered by  $X_{\text{unr}}(L)$  via (34). We write

$$X_{\text{nr}}(L, \sigma) = \{\chi \in X_{\text{nr}}(L) : \sigma \otimes \chi \cong \sigma\}.$$

This is a finite subgroup of  $X_{\text{unr}}(L)$ . The map (34) induces an isomorphism of algebraic varieties  $X_{\text{nr}}(L)/X_{\text{nr}}(L, \sigma) \rightarrow T_\mathfrak{s}$ .

The group  $W(G, L) = N_G(L)/L$  acts on  $\text{Irr}(L)$  by

$$(35) \quad (gL \cdot \pi)(l) = \pi(glg^{-1}).$$

(The representation  $gL \cdot \pi$  is only determined up to isomorphism.) This action stabilizes  $X_{\text{nr}}(L)$ , the unitary representations in  $\text{Irr}(L)$  and the supercuspidal  $L$ -representations. Let  $W_\mathfrak{s}$  be the stabilizer of  $T_\mathfrak{s}$  in  $N_G(L)/L$ . This group will play the same role as  $W\Gamma$  did in Section 1. The theory of the Bernstein centre [BeDe, Théorème 2.13] says that the centre of  $\mathcal{H}(G, K_\mathfrak{s})^\mathfrak{s}$  is naturally isomorphic with  $\mathcal{O}(T_\mathfrak{s})^{W_\mathfrak{s}} = \mathcal{O}(T_\mathfrak{s}/W_\mathfrak{s})$ .

It will be convenient to lift everything from  $T_\mathfrak{s}$  to  $X_{\text{nr}}(L)$ . However,  $W_\mathfrak{s}$  does not act naturally on  $X_{\text{nr}}(L)$ . To overcome this and similar issues, we need the following lemma.

**Lemma 2.1.** *Let  $p : T' \rightarrow T$  be a surjection between complex tori, with finite kernel  $K = \ker p$ . Let  $\Gamma$  be a finite group acting on  $T$  by automorphisms of algebraic varieties (so  $\Gamma$  need not fix  $1 \in T$ ). Then there exists a canonical short exact sequence*

$$1 \rightarrow K \rightarrow \Gamma' \rightarrow \Gamma \rightarrow 1$$

*and a canonical action of  $\Gamma'$  on  $T'$  which extends the multiplication action of  $K$  on  $T'$  and lifts the action of  $\Gamma$  on  $T$ .*

*Proof.* Let  $X$  be the character lattice of  $T$ . Then  $\mathcal{O}(T) \cong \mathbb{C}[X]$  and  $\Gamma$  acts on  $\mathcal{O}(T)$  by  $(\gamma \cdot f)(t) = f(\gamma^{-1}t)$ . Since

$$\mathcal{O}(T)^\times = \{z\theta_x : z \in \mathbb{C}^\times, x \in X\} \cong \mathbb{C}^\times \times X,$$

$\Gamma$  also acts naturally on  $X \cong \mathcal{O}(T)^\times / \mathbb{C}^\times$ . Let us denote this action by  $l_\gamma : X \rightarrow X$ . Notice that it defines an action of  $\Gamma$  on  $T = \text{Hom}_{\mathbb{Z}}(X, \mathbb{C}^\times)$  by algebraic group automorphisms. The given action on  $\mathcal{O}(T)$  can now be written as

$$\gamma(z\theta_x) = zz_\gamma^{-1}(l_\gamma(x))\theta_{l_\gamma(x)},$$

for a unique  $z_\gamma \in T$ . Consequently the original action of  $\Gamma$  on  $T$  can be expressed as

$$(36) \quad \gamma(t) = z_\gamma l_\gamma(t).$$

The character lattice  $X'$  of  $T'$  contains  $X$  with finite index  $|K|$ , so  $l_\gamma$  induces a canonical linear action of  $\Gamma$  on  $X'$ , also denoted  $l$ . For every  $\gamma \in \Gamma$  we choose a  $z'_\gamma \in p^{-1}(z_\gamma)$ , and we define

$$\phi_\gamma : T' \rightarrow T', \quad \phi_\gamma(t') = z'_\gamma l_\gamma(t').$$

Clearly  $\phi_\gamma$  is a lift of (36), so for every  $\gamma, \gamma' \in \Gamma$  there exists a unique  $z'_{\gamma, \gamma'} \in K$  with

$$(37) \quad \phi_\gamma \circ \phi_{\gamma'} \circ \phi_{\gamma, \gamma'}^{-1}(t') = z'_{\gamma, \gamma'} t' \quad \forall t' \in T'.$$

Let  $\Gamma'$  be the subgroup of  $\text{Aut}(T')$  generated by the  $\phi_\gamma$  ( $\gamma \in \Gamma$ ) and  $K$ . Then (37) gives a canonical isomorphism  $\Gamma'/K \cong \Gamma$ .

The only unnatural steps in the above argument are the choices of the  $z'_\gamma$ . Different choices would lead to different  $z'_{\gamma, \gamma'}$  in (37), but to the same group  $\Gamma'$ . Hence  $\Gamma'$  is canonically determined by the data  $T, T'$  and  $\Gamma$ .  $\square$

Next we recall the Plancherel isomorphism for  $\mathcal{S}(G)^s$ , as discovered by Harish-Chandra and worked out by Waldspurger. As induction data for  $G$  we take quadruples  $(P, M, \omega, \chi)$ , where

- $P$  is a parabolic subgroup of  $G$  with a Levi factor  $M$ ;
- $(\omega, V_\omega) \in \text{Irr}_{L^2}(M)$ , the set of (isomorphism classes of) irreducible smooth square-integrable modulo centre representations of  $M$ ;
- $\chi \in X_{\text{unr}}(M)$ .

To such a datum we associate the smooth  $G$ -representation  $I_P^G(\omega \otimes \chi)$ , where  $I_P^G$  denotes normalized parabolic induction. The  $M$ -invariant inner product on  $(\omega \otimes \chi, V_\omega)$  induces a  $G$ -invariant inner product on  $I_P^G(V_\omega)$ , so  $I_P^G(\omega \otimes \chi)$  is pre-unitary [Cas, Proposition 3.1.4]. However,  $I_P^G(V_\omega)$  is only complete with respect to the associated metric if  $\dim(I_P^G(V_\omega))$  is finite.

Let  $(\check{\omega}, \check{V}_\omega)$  be the smooth contragredient of  $\omega$  and put

$$\mathfrak{L}(\omega, P) = I_{P \times P}^{G \times G}(\omega \otimes \check{\omega}) = I_P^G(\omega) \otimes I_P^G(\check{\omega}).$$



Since  $I_P^G(\tilde{\omega})$  can be identified with the smooth contragredient of  $I_P^G(\omega)$  [Cas, Proposition 3.1.2],  $\mathfrak{L}(\omega, P)$  can be regarded as the algebra of finite rank linear operators on  $I_P^G(V_\omega)$ . Notice that for every  $\chi \in X_{\text{nr}}(M)$  we can identify  $\mathfrak{L}(\omega \otimes \chi, P)$  with  $\mathfrak{L}(\omega, P)$  as algebras. The inner product on  $I_P^G(V_\omega)$  induces a  $*$ -operation on this algebra. That makes  $\mathcal{O}(X_{\text{nr}}(M)) \otimes \mathfrak{L}(\omega, P)$  to a  $*$ -algebra with

$$f^*(\chi) = f(\check{\chi})^*.$$

There is a natural  $*$ -homomorphism

$$(38) \quad \begin{aligned} \mathcal{H}(G) &\rightarrow \mathcal{O}(X_{\text{nr}}(M)) \otimes \mathfrak{L}(\omega, P), \\ f &\mapsto (\chi \mapsto I_P^G(\omega \otimes \chi)(f)). \end{aligned}$$

We put  $T_\omega = \{\omega \otimes \chi : \chi \in X_{\text{nr}}(M)\}$  and we record the covering map

$$(39) \quad X_{\text{nr}}(M) \rightarrow T_\omega : \chi \mapsto \omega \otimes \chi.$$

The group

$$X_{\text{nr}}(M, \omega) = \{\chi \in X_{\text{nr}}(M) : \omega \otimes \chi \cong \omega\}$$

is finite, because all its elements must be trivial on  $Z(M)$ . All the fibres of (39) are isomorphic to  $X_{\text{nr}}(M, \omega)$ .

For every  $k \in X_{\text{nr}}(M, \omega)$  there exists a unitary  $M$ -intertwiner  $\omega \rightarrow \omega \otimes k$ , unique up to scalars. The same map  $V_\omega \rightarrow V_\omega$  also intertwines  $\omega \otimes \chi$  with  $\omega \otimes \chi k$ , for any  $\chi \in X_{\text{nr}}(M)$ . Applying  $I_P^G$ , we get a family a  $G$ -intertwiners

$$(40) \quad \pi(k, \omega, \chi) : I_P^G(\omega \otimes \chi) \rightarrow I_P^G(\omega \otimes \chi k),$$

independent of  $\chi$  and unitary when  $\chi \in X_{\text{unr}}(M)$ . Let

$$\tilde{\pi}(k, \omega, \chi) : I_P^G(\tilde{\omega} \otimes \check{\chi}) \rightarrow I_P^G(\tilde{\omega} \otimes \check{\chi} k)$$

be the inverse transpose  $\pi(k, \omega, \chi)$ . Since  $\pi(k, \omega, \chi)$  is unique up to scalars,

$$(41) \quad I(k, \omega, \chi) := \pi(k, \omega, \chi) \otimes \tilde{\pi}(k, \omega, \chi) \in \text{Hom}_{G \times G}(\mathfrak{L}(\omega \otimes \chi, P), \mathfrak{L}(\omega \otimes \chi k, P))$$

is canonical. Moreover it is unitary for  $\chi \in X_{\text{unr}}(M)$  and independent of  $\chi$  as map between vector spaces.

Let  $W(T_\omega)$  be the stabilizer of  $T_\omega$  in  $W(G, M) = N_G(M)/M$ , with respect to the action on  $\text{Irr}(M)$  as in (35). Then  $W(T_\omega)$  acts naturally on  $T_\omega$ . From Lemma 2.1 we get a group extension

$$(42) \quad 1 \rightarrow X_{\text{nr}}(M, \omega) \rightarrow W'(T_\omega) \rightarrow W(T_\omega) \rightarrow 1$$

and an action of  $W'(T_\omega)$  on  $X_{\text{nr}}(M)$  compatible with the covering (39). In [Wal] the representations  $\omega \otimes \chi$  and  $\omega \otimes \chi k$  are often not distinguished. The introduction of  $W'(T_\omega)$  and of the  $I(k, \omega, \chi)$  allows us to compare  $I_P^G(\omega \otimes \chi)$  and  $I_P^G(\omega \otimes \chi k)$  in a systematic way. From [Wal, §VI.1] one can see that actually our setup is just another way to keep track of all the ingredients of [Wal].

The following results are proven in [Wal, Paragraphe V]. For  $w' \in W'(T_\omega)$  there exist unitary  $G$ -intertwining operators

$$(43) \quad \pi(w', \omega, \chi) : I_P^G(\omega \otimes \chi) \rightarrow I_P^G(\omega \otimes w'(\chi)) \quad \chi \in X_{\text{unr}}(M),$$

unique up to scalars. These give canonical unitary intertwiners

$$(44) \quad I(w', \omega, \chi) = \pi(w', \omega, \chi) \otimes \tilde{\pi}(w', \omega, \chi) \in \text{Hom}_{G \times G}(\mathfrak{L}(\omega \otimes \chi, P), \mathfrak{L}(\omega \otimes w'(\chi), P))$$

with the following properties [Wal, Lemme V.3.1]:

- as functions of  $\chi$ ,  $\pi(w', \omega, \chi)$  and  $I(w', \omega, \chi)$  are continuous with respect to the Zariski topology on the real algebraic variety  $X_{\text{unr}}(M)$ ;
- $I(w'_2, \omega, w'_1(\chi)) \circ I(w'_1, \omega, \chi) = I(w'_2 w'_1, \omega, \chi)$  for  $w'_1, w'_2 \in W'(T_\omega)$ .

We remark that  $I(w', \omega, \chi)$  is called  ${}^{\circ}c_{P|P}(w', \omega \otimes \chi)$  in [Wal]. These intertwining operators give rise to an action of  $W'(T_\omega)$  on the algebra

$$C^\infty(X_{\text{unr}}(M)) \otimes \mathfrak{L}(\omega, P) \quad \text{by} \quad (w' \cdot f)(w' \chi) = I(w', \omega, \chi) f(\chi).$$

We fix a parabolic subgroup  $P_L$  with Levi factor  $L$ , and we recall that  $\mathfrak{s} = [L, \sigma]_G$ . To study representations in the Bernstein block  $\text{Rep}(G)^\mathfrak{s}$ , it suffices to consider induction data such that  $P \supset P_L, M \supset L$  and the cuspidal support of  $\omega$  lies in  $[L, \sigma]_M$ . Then  $W(T_\omega)$  can be regarded as a subgroup of  $W_\mathfrak{s}$ .

Choose representatives for the  $G$ -association classes of parabolic subgroups  $P$  containing  $P_L$ . Notice that every such  $P$  has a unique Levi factor  $M$  containing  $L$ . We also choose representatives  $\omega$  for the action of  $X_{\text{unr}}(M)$  on  $\text{Irr}_{L^2}(M) \cap \text{Irr}(M)^{\mathfrak{s}_M}$ , where  $\mathfrak{s}_M = [L, \sigma]_M$ . We denote the resulting set of representative triples by  $(P, M, \omega)/\sim$ . Harish-Chandra established the following Plancherel isomorphism, see [SZ, Theorem 8.9] for an alternative proof.

**Theorem 2.2.** [Wal, Théorème VII.2.5]

*The maps (38) induces isomorphisms of topological  $*$ -algebras*

$$\begin{aligned} \mathcal{S}(G)^\mathfrak{s} &\rightarrow \bigoplus_{(P, M, \omega)/\sim} (C^\infty(X_{\text{unr}}(M)) \otimes \mathfrak{L}(\omega, P))^{W'(T_\omega)}, \\ \mathcal{S}(G, K_\mathfrak{s})^\mathfrak{s} &\rightarrow \bigoplus_{(P, M, \omega)/\sim} \left( C^\infty(X_{\text{unr}}(M)) \otimes \text{End}_{\mathbb{C}}(I_P^G(V_\omega)^{K_\mathfrak{s}}) \right)^{W'(T_\omega)}. \end{aligned}$$

Plymen [Ply] showed that Theorem 2.2 has a natural extension to  $C^*$ -algebras. Let  $\mathfrak{H}(\omega \otimes \chi, P)$  be the Hilbert space completion of  $I_P^G(V_{\omega \otimes \chi}) = I_P^G(V_\omega)$  and let  $\mathfrak{K}(\omega \otimes \chi, P)$  be the  $C^*$ -algebra of compact operators on  $\mathfrak{H}(\omega \otimes \chi, P)$ .

**Theorem 2.3.** *The maps (38) induces isomorphisms of  $C^*$ -algebras*

$$\begin{aligned} C_r^*(G)^\mathfrak{s} &\rightarrow \bigoplus_{(P, M, \omega)/\sim} C(X_{\text{unr}}(M); \mathfrak{K}(\omega, P))^{W'(T_\omega)}, \\ C_r^*(G, K_\mathfrak{s})^\mathfrak{s} &\rightarrow \bigoplus_{(P, M, \omega)/\sim} \left( C(X_{\text{unr}}(M)) \otimes \text{End}_{\mathbb{C}}(I_P^G(V_\omega)^{K_\mathfrak{s}}) \right)^{W'(T_\omega)}. \end{aligned}$$

*Proof.* First we note that we have intertwining operators associated to the group  $W'(T_\omega)$ , instead of  $W(T_\omega)$  in [Ply, Wal]. The reason for this is explained after (42). In view of Theorem 2.2, it only remains to prove that completing with respect to the operator norm of  $C_r^*(G)$  boils down to replacing  $C^\infty(X_{\text{unr}}(M)) \otimes \mathfrak{L}(\omega, P)$  by  $C(X_{\text{unr}}(M); \mathfrak{K}(\omega, P))$ . This is shown in [Ply, Theorem 2.5].  $\square$

### 3. COMPARISON OF COMPLETIONS

In this section we will first formulate a long list of conditions for the objects we want to compare. Assuming these conditions, we will prove a comparison theorem. In the next sections we will check that these conditions are fulfilled in cases of interest. We keep the notations from the previous section.

**Condition 3.1.** For every parabolic subgroup  $P$  with  $P_L \subset P \subset G$  and Levi factor  $M \supset L$ , an algebra  $\mathcal{H}^M$  and a Morita equivalence

$$\Phi_M : \text{Rep}(M)^{\mathfrak{s}_M} \rightarrow \text{Mod}(\mathcal{H}^M)$$

are given.

When  $P' \supset P$  is another such parabolic subgroup, an algebra injection  $\lambda_{MM'} : \mathcal{H}^M \rightarrow \mathcal{H}^{M'}$  is given, such that:

- (i) The following diagram commutes:

$$\begin{array}{ccc} \text{Rep}(M')^{\mathfrak{s}_{M'}} & \xrightarrow{\Phi_{M'}} & \text{Mod}(\mathcal{H}^{M'}) \\ \uparrow I_{P \cap M'}^{M'} & \text{ind}_{\lambda_{MM'}(\mathcal{H}^M)}^{\mathcal{H}^{M'}} & \uparrow \\ \text{Rep}(M)^{\mathfrak{s}_M} & \xrightarrow{\Phi_M} & \text{Mod}(\mathcal{H}^M) \end{array}$$

- (ii) Let  $J_P^G$  be the normalized Jacquet restriction functor and let  $\overline{P}$  be the parabolic subgroup of  $G$  which has Levi factor  $M$  and is opposite to  $P$ . Let  $\text{pr}_{\mathfrak{s}_M} : \text{Rep}(M) \rightarrow \text{Rep}(M)^{\mathfrak{s}_M}$  be the projection coming from the Bernstein decomposition for  $M$ . The following diagram commutes:

$$\begin{array}{ccc} \text{Rep}(M')^{\mathfrak{s}_{M'}} & \xrightarrow{\Phi_{M'}} & \text{Mod}(\mathcal{H}^{M'}) \\ \downarrow \text{pr}_{\mathfrak{s}_M} \circ J_{\overline{P} \cap M'}^{M'} & & \downarrow \text{Res}_{\lambda_{MM'}(\mathcal{H}^M)}^{\mathcal{H}^{M'}} \\ \text{Rep}(M)^{\mathfrak{s}_M} & \xrightarrow{\Phi_M} & \text{Mod}(\mathcal{H}^M) \end{array}$$

- (iii) If  $P \subset P' \subset P'' \subset G$ , then  $\lambda_{MM''} = \lambda_{M'M''} \circ \lambda_{MM'}$ .

The Conditions 3.1 are quite general, in the sense that they do not involve the structure of the algebras  $\mathcal{H}^M$ . We will see later that in many cases these conditions hold already by abstract functoriality principles.

The next series of conditions is much more specific though. For  $P = MU$ , let  $R(M, L)$  be the set of roots of  $M$  with respect to the maximal  $F$ -split torus in the centre of  $L$ . This is a root system when  $L$  is a minimal  $F$ -Levi subgroup of  $G$ . In general it is only an orthogonal projection of such a root system. For  $P \supset P_L$  we define the set of positive roots as  $R^+(M, L) = R(M \cap P_L, L)$ .

**Condition 3.2.** Assume Condition 3.1.

- (i)  $\mathcal{H}^G$  (or  $(\mathcal{H}^G)^{\text{op}}$ ) is an extended affine Hecke algebra  $\mathcal{H}(\mathcal{R}, q) \rtimes \Gamma$ .
- (ii) All the  $\mathcal{H}^M$  (or all the  $(\mathcal{H}^M)^{\text{op}}$ ) are parabolic subalgebras and the  $\lambda_{MM'}$  are inclusions of parabolic subalgebras.
- (iii) The bijection

$$\Phi_L : X_{\text{nr}}(L)/X_{\text{nr}}(L, \sigma) \cong \text{Irr}(L)^{\mathfrak{s}_L} \rightarrow \text{Irr}(\mathcal{H}^L) \cong T$$

induces an injection from the coroots  $R^\vee$  for  $\mathcal{H}(\mathcal{R}, q)$  to  $R(G, L)_{\text{red}}$ , which preserves positivity.

- (iv) Suppose that  $\mathbb{Q}Q^\vee = \mathbb{Q}R^\vee \cap \mathbb{Q}R(M, L)$  via part (3). Then  $\mathcal{H}^M = \mathcal{H}^Q \rtimes \Gamma_M$  for some  $\Gamma_M \subset \Gamma$ . If moreover  $\mathbb{Q}R(M, L) = \mathbb{Q}Q^\vee$ , then  $\Gamma_M$  satisfies Condition 1.2 for  $Q$ .

These conditions allow us to draw some consequences about temperedness and discrete series. The below argument is similar to [Hei2], but we put it in a more general framework.

**Proposition 3.3.** Assume Conditions 3.1 and 3.2.

- (a)  $\Phi_M$  restricts to an equivalence between the category of finite length tempered representations in  $\text{Rep}(M)^{\mathfrak{s}_M}$  and the category of finite dimensional tempered  $\mathcal{H}^M$ -modules.

(b) Suppose that  $\mathbb{Q}R(M, L) \subset \mathbb{Q}R^\vee$  via Condition 3.2.(iii). Then  $\Phi_M$  gives a bijection between  $\text{Irr}_{L^2}(M)^{\mathfrak{s}_M}$  and  $\text{Irr}_{L^2}(\mathcal{H}^M)$ .

If  $P' = M'U' \supsetneq P$  and  $\mathbb{Q}R(M', L) \cap \mathbb{Q}R^\vee = \mathbb{Q}(M, L)$ , then  $\text{Rep}(M')^{\mathfrak{s}_{M'}}$  contains no square-integrable modulo centre representations (but  $\text{Irr}_{L^2}(\mathcal{H}^{M'})$  can still be nonempty).

*Proof.* (a) Since every irreducible  $\mathcal{H}^M$ -module has finite dimension,  $\Phi_M$  restricts to an equivalence between finite length representations on the one hand, and the finite dimensional modules on the other hand.

Let  $M$  be such that  $\mathbb{Q}R(M, L) \subset \mathbb{Q}R^\vee$  via Condition 3.2.(iii). Let  $\pi \in \text{Rep}(M)^{\mathfrak{s}_M}$  be of finite length. By [Wal, Proposition III.2.2] it is tempered if and only if: for every parabolic  $P_1 = M_1U_1 \supset P_L$  with  $M \supset M_1 \supset L$  and every  $Z(M_1)^\circ$ -weight  $\chi$  of  $J_{P_1 \cap M}^M \pi$ ,

$$|\chi| \in \overline{+a_{P_1 \cap M}^M} := \left\{ \sum_{\alpha \in \Delta(P_1 \cap M)} c_\alpha \alpha : c_\alpha \geq 0 \right\}.$$

Here  $\Delta(P_1 \cap M)$  is the subset of simple roots determined by  $P_1$ . Let  $\overline{P_L}$  be the parabolic subgroup opposite to  $P_L$ . Then  $\overline{P_L} \cap M$  is opposite to  $P$ . The above condition is equivalent to:

$$|\chi| \in \overline{+a_{\overline{P_L} \cap M}^M} := \left\{ \sum_{\alpha \in \Delta(P_L \cap M)} c_\alpha \alpha : c_\alpha \leq 0 \right\}$$

for every  $Z(L)^\circ$ -weight  $\chi$  of  $J_{M \cap \overline{P_L}}^M \pi$ . With our assumptions we can translate this to a statement about  $\Phi_M(\pi)$ . Namely: all  $\mathcal{H}^L$ -weights  $t$  of  $\text{Res}_{\lambda_{LM}}^{\mathcal{H}^M}(\Phi_M(\pi))$  satisfy  $|t| \in \mathfrak{a}^-$ . But that is exactly the definition of temperedness of  $\Phi_M(\pi)$ . Thus  $\pi$  is tempered if and only if  $\Phi_M(\pi)$  is tempered.

Let  $P' = M'U' \supset P_L$  and define, using Condition 3.2.(iii),  $M$  by  $\mathbb{Q}R(M', L) \cap \mathbb{Q}R^\vee = \mathbb{Q}R(M, L)$ . We will derive the desired result for  $M'$  from that for  $M$ . By Condition 3.2.(iv),  $\mathcal{H}^{M'} = \mathcal{H}^Q \rtimes \Gamma_{M'}$  and  $\mathcal{H}^M = \mathcal{H}^Q \rtimes \Gamma_M$ , where  $\Gamma_{M'} \supset \Gamma_M$  and  $\lambda_{MM'}$  is just the inclusion. The cone  $T^- \subset T_{\text{rs}}$  is the same for  $\mathcal{H}^M, \mathcal{H}^Q$  and  $\mathcal{H}^{M'}$ . For any finite dimensional  $\mathcal{H}^M$ -module  $V$ ,

$$(45) \quad \text{Wt}(\text{ind}_{\mathcal{H}^M}^{\mathcal{H}^{M'}}(V)) = \{\gamma(t) : t \in \text{Wt}(V), \gamma \in \Gamma_{M'}\}.$$

Since  $\Gamma_{M'}$  preserves  $T^-$ ,  $\text{ind}_{\mathcal{H}^M}^{\mathcal{H}^{M'}}(V)$  is tempered if and only if  $V$  is tempered. Similarly, a finite dimensional  $\mathcal{H}^{M'}$ -module  $V'$  is tempered if and only if  $\text{Res}_{\mathcal{H}^M}^{\mathcal{H}^{M'}}(V')$  is tempered. We note also that

$$(46) \quad \text{ind}_{\mathcal{H}^M}^{\mathcal{H}^{M'}} \text{Res}_{\mathcal{H}^M}^{\mathcal{H}^{M'}}(V') \cong \mathbb{C}[\Gamma_{M'}] \otimes_{\mathbb{C}[\Gamma_M]} V' \cong \mathbb{C}[\Gamma_{M'}/\Gamma_M] \otimes_{\mathbb{C}} V',$$

a  $\mathcal{H}^Q \rtimes \Gamma_{M'}$ -module for the diagonal action. Then (46) contains  $V'$  as the direct summand  $\mathbb{C}[\Gamma_{M'}/\Gamma_M] \otimes_{\mathbb{C}} V'$ , and the restriction of (46) to  $\mathcal{H}^M$  is a direct sum of copies of  $\text{Res}_{\mathcal{H}^M}^{\mathcal{H}^{M'}}(V')$ .

We recall from [Ren, Lemme VII.2.2] that  $I_{P \cap M'}^{M'}$  always preserves temperedness. (On the other hand, in general  $J_{P \cap M'}^{M'}$  does not, which causes some of our problems.) Consider a finite dimensional tempered module  $V' \in \text{Mod}(\mathcal{H}^{M'})$ . By what we showed above,

$$I_{P \cap M'}^{M'} \circ \Phi_M^{-1} \circ \text{Res}_{\mathcal{H}^M}^{\mathcal{H}^{M'}}(V') = I_{P \cap M'}^{M'} \circ \text{pr}_{\mathfrak{s}_M} \circ J_{\overline{P} \cap M'}^{M'} \circ \Phi_{M'}^{-1}(V')$$

is tempered. By the analogue of (46), this representation contains  $\Phi_{M'}^{-1}(V')$  as a direct summand, so that is tempered as well. Hence  $\Phi_{M'}^{-1}$  preserves temperedness.

For any finite length  $\pi \in \text{Rep}(M)^{s_M}$ ,  $I_{P \cap M'}^{M'}(\pi)$  is not tempered when  $\pi$  is not, because  $\pi$  is a quotient of  $J_{P \cap M'}^{M'} I_{P \cap M'}^{M'}(\pi)$ . Hence  $I_{P \cap M'}^{M'}(\pi)$  is tempered if and only if  $\pi$  is tempered.

Condition 3.2 guarantees that the properties described in and after (46) hold for  $\text{Rep}(M)^{s_M}$  and  $\text{Rep}(M')^{s_{M'}}$ . Thus any finite length  $\pi' \in \text{Rep}(M')^{s_{M'}}$  is a direct summand of

$$(47) \quad I_{P \cap M'}^{M'} \circ \text{pr}_{s_M} \circ J_{P \cap M'}^{M'}(\pi'),$$

and applying  $\text{pr}_{s_M} \circ J_{P \cap M'}^{M'}$  to the latter representation gives a finite sum of copies of  $\text{pr}_{s_M} \circ J_{P \cap M'}^{M'}(\pi')$ . But the temperedness of  $\pi$  is defined via  $J_{P \cap M'}^{M'}(\pi')$ . The projection  $\text{pr}_{s_M} : \text{Rep}(M) \rightarrow \text{Rep}(M)^{s_M}$  is harmless here, because all the components of  $J_{P \cap M'}^{M'}(\pi')$  in other Bernstein blocks of  $\text{Rep}(M)$  are  $M'$ -conjugate to  $\text{pr}_{s_M} \circ J_{P \cap M'}^{M'}(\pi')$ .

It follows that  $\pi'$  is tempered if and only (47) is tempered, if and only if  $\text{pr}_{s_M} \circ J_{P \cap M'}^{M'}(\pi')$  is tempered. Assume now that  $\pi'$  is indeed tempered. Then also

$$\text{ind}_{\mathcal{H}_M}^{\mathcal{H}_{M'}} \circ \Phi_M \circ \text{pr}_{s_M} \circ J_{P \cap M'}^{M'}(\pi') = \text{ind}_{\mathcal{H}_M}^{\mathcal{H}_{M'}} \circ \text{Res}_{\mathcal{H}_M}^{\mathcal{H}_{M'}} \circ \Phi_{M'}(\pi')$$

is tempered. As this module contains  $\Phi_{M'}(\pi')$ , we can conclude that  $\Phi_{M'}$  preserves temperedness.

(b) For  $M$  as above this can be shown in the same way, only now with

$${}^+a_{P_1}^G = \left\{ \sum_{\alpha \in \Delta(P_1)} c_\alpha \alpha : c_\alpha > 0 \right\}.$$

For  $M'$  as above (47) shows essentially that every representation in  $\text{Rep}(M')^{s_{M'}}$  is obtained by parabolic induction from a proper Levi subgroup. Thus  $\text{Rep}(M')^{s_{M'}}$  does not contain any square-integrable modulo centre representations.  $\square$

Notice that Proposition 3.3 also says something about unitarity, because irreducible tempered modules are unitary. However, at this stage we do not know whether  $\Phi_G$  preserves unitarity for nontempered modules.

**Theorem 3.4.** *Under the Conditions 3.1, 3.2 and 1.2,  $\Phi_G : \text{Rep}(G)^s \rightarrow \text{Mod}(\mathcal{H}^G)$  induces Morita equivalences*

$$\mathcal{S}(G)^s \sim_M \mathcal{S}(\mathcal{R}, q) \rtimes \Gamma \quad \text{and} \quad C_r^*(G)^s \sim_M C_r^*(\mathcal{R}, q) \rtimes \Gamma.$$

*Proof.* In Proposition 1.5 we described the Plancherel isomorphism for the Schwartz completion of an affine Hecke algebra, in terms of the following data:

- the set of parabolic subalgebras  $\mathcal{H}^Q \rtimes \Gamma_Q$  of  $\mathcal{H} \rtimes \Gamma$ , up to  $\Gamma W$ -equivalence,
- the tori  $T_{\text{un}}^Q$ ,
- the sets  $\text{Irr}_{L^2}(\mathcal{H}^Q \rtimes \Gamma_Q)$ , up to the actions of  $T_{\text{un}}^Q$  and  $W\Gamma(Q, Q)$ ,
- the groupoid  $\mathcal{G}$ ,
- the intertwining operators  $I(g, Q, \sigma, t)$  for  $g \in \mathcal{G}_{Q, \sigma}$ .

These data depend mainly on the categories  $\text{Mod}(\mathcal{H}^Q \rtimes \Gamma_Q)$ . In Condition 3.2 we included the possibility that not the  $\mathcal{H}^M$ , but the  $(\mathcal{H}^M)^{\text{op}}$  are affine Hecke algebras, so that  $\Phi_M$  becomes an equivalence between  $\text{Rep}(G)^s$  and  $\text{Mod}((\mathcal{H}^Q \rtimes \Gamma_Q)^{\text{op}})$ . Then we use Lemma 1.6 to describe the Plancherel isomorphism of  $\mathcal{S}(\mathcal{R}, q) \rtimes \Gamma$  in terms of right modules of its subalgebras  $\mathcal{H}^Q \rtimes \Gamma_Q$ , that is in terms of the categories

$\text{Mod}(\mathcal{H}^M)$ . With this in mind, it suffices to consider the case where each  $\mathcal{H}^M$  is an (extended) affine Hecke algebra.

On the other hand, in Theorem 2.2 the Plancherel isomorphism for  $\mathcal{S}(G)^s$  was formulated in terms of:

- the set of parabolic subgroups  $P \supset P_L$ , up to conjugation by  $W_s$ ,
- the tori  $X_{\text{unr}}(M)$ ,
- the sets  $\text{Irr}_{L^2}(M)^{s_M}$ , up to the actions of  $X_{\text{unr}}(M)$  and  $\text{Stab}_{W_s}(M)$ ,
- the groups  $W'(T_\omega)$ ,
- the intertwining operators  $I(w', \omega \otimes \chi)$  for  $w' \in W'(T_\omega)$ .

We will compare these two data sets, and manipulate them until we get a nice bijection from one side to the other.

By Proposition 3.3.b only the  $P$  with  $\mathbb{Q}R(M, L) \subset \mathbb{Q}R^\vee$  occur in the Plancherel isomorphism, since for the other  $P$  the set  $\text{Irr}_{L^2}(M)^{s_M}$  is empty. Given  $Q \subset \Delta$ , we define  $\Gamma_Q$  as  $\Gamma_M$ , where  $\mathbb{Q}R(M, L) = \mathbb{Q}Q^\vee$ .

By Condition 3.2.(iv) there is a canonical bijection from the parabolic subgroups  $P = MU_P$  with  $P \supset P_L, M \supset L$  and  $\mathbb{Q}R(M, L) \subset \mathbb{Q}R^\vee$  to the parabolic subalgebras  $\mathcal{H}^Q$  of  $\mathcal{H}^G$ . Two such Levi subgroups  $M \subset G$  are  $G$ -conjugate if and only if the two subsets  $I_P^G(\text{Rep}(M)^{s_M})$  coincide. By Condition 3.1.(1) this means precisely that two subsets  $\text{ind}_{\lambda_{MG}(\mathcal{H}^M)}^{\mathcal{H}^G}(\text{Mod}(\mathcal{H}^M))$  of  $\text{Mod}(\mathcal{H}^G)$  coincide. By Theorem 1.1 that happens if and only if the two  $\mathcal{H}^M$  are  $\Gamma W$ -equivalent. Thus we can pick of representatives for such  $P$  modulo  $G$ -conjugacy, and then the corresponding  $\mathcal{H}^M$  form representatives for  $\Gamma W$ -equivalence classes of parabolic subalgebras  $\mathcal{H}^Q$  of  $\mathcal{H}^G$ .

By Proposition 3.3  $\Phi_M$  gives a bijection between  $\text{Irr}_{L^2}(M)^{s_M}$  and  $\text{Irr}_{L^2}(\mathcal{H}^M)$ . Upon parabolic induction, every  $X_{\text{unr}}(M)$ -orbit in  $\text{Irr}_{L^2}(M)^{s_M}$  (resp. every  $T_{\text{unr}}^Q$ -orbit in  $\text{Irr}_{L^2}(\mathcal{H}^M)$ ) gives rise to a family of representations in  $\text{Rep}(G)^s$  (resp. in  $\text{Mod}(\mathcal{H}^G)$ ). From Theorem 2.2 we see that  $I_P^G(\omega)$  and  $I_P^G(\omega')$  belong to the same such family if and only if  $\omega' = w(\omega \otimes \chi)$  for some  $w \in \text{Stab}_{W_s}(M)$  and  $\chi \in X_{\text{unr}}(M)$ . Similarly, by 3.2.(ii) and Proposition 1.5

$$\text{ind}_{\lambda_{MG}(\mathcal{H}^M)}^{\mathcal{H}^G}(\sigma) \quad \text{and} \quad \text{ind}_{\lambda_{MG}(\mathcal{H}^M)}^{\mathcal{H}^G}(\sigma')$$

belong to the same family in  $\text{Mod}(\mathcal{H}^G)$  if and only if  $\sigma' = g(\sigma \circ \phi_t)$  for some  $g \in \mathcal{G}_{QQ}$  and  $t \in T_{\text{unr}}^Q$ .

Applying  $\Phi_G$  and Condition 3.1.(i), we see that the respective equivalence relations on  $\text{Irr}_{L^2}(M)^{s_M}$  and  $\text{Irr}_{L^2}(\mathcal{H}^M)$  agree via  $\Phi_M$ .

Let the representatives  $(Q, \sigma)/\sim$  be as in (28) Let  $(P, M, \omega)/\sim$  be its image under Condition 3.2.(iv) and the  $\Phi_M^{-1}$ . Then  $(P, M, \omega)/\sim$  is a set of representatives as in Theorem 2.2

We turn to the tori. For the reductive groups we have

$$\text{Irr}(L)^{s_L} = \{\sigma \otimes \chi : \chi \in X_{\text{nr}}(L)\} \cong X_{\text{nr}}(L)/X_{\text{nr}}(L, \sigma).$$

By Condition 3.2.(ii)  $\Phi_L$  gives a bijection  $\text{Irr}(L)^{s_L} \rightarrow \text{Irr}(\mathcal{H}^L) = T$ . This lifts to a surjection

$$(48) \quad \Phi_{\text{nr}} : X_{\text{nr}}(L) \rightarrow T$$

with kernel  $X_{\text{nr}}(L, \sigma)$ . For every  $Q \subset \Delta$  we defined the subtorus

$$T^Q = \{t \in T : t(x) = 1 \ \forall x \in \mathbb{Q}Q \cap X\}$$

of  $T$ . Using Condition 3.2.(iv) we can write

$$X_{\text{nr}}(M) = \{\chi \in X_{\text{nr}}(L) : \chi = 1 \text{ on } \mathbb{Q}Q \cap X^*(X_{\text{nr}}(L))\},$$

which shows that  $X_{\text{nr}}(M)$  is the preimage of  $T^P$  under (48).

The kernel of  $\Phi_{\text{nr}} : X_{\text{nr}}(M) \rightarrow T^Q$  is

$$X_{\text{nr}}(M, \sigma) := X_{\text{nr}}(L, \sigma) \cap X_{\text{nr}}(M).$$

Then  $X_{\text{nr}}(M, \sigma)$  acts on  $X_{\text{nr}}(M)$  by translations and  $\mathcal{G}_{Q, \sigma}$  acts on  $T^Q \cong X_{\text{nr}}(M)/X_{\text{nr}}(M, \sigma)$ . By Lemma 2.1 there exists a canonical short exact sequence

$$(49) \quad 1 \rightarrow X_{\text{nr}}(M, \sigma) \rightarrow \mathcal{G}'_{Q, \sigma} \rightarrow \mathcal{G}_{Q, \sigma} \rightarrow 1,$$

such that the action of  $\mathcal{G}'_{Q, \sigma}$  on  $X_{\text{nr}}(M)$  lifts that of  $\mathcal{G}_{Q, \sigma}$  on  $T^Q$ . For  $\omega = \Phi_M^{-1}(\sigma)$  Condition 3.1.(ii) implies that

$$(50) \quad \text{Res}_{\lambda_{LM}(\mathcal{H}^L)}^{\mathcal{H}^M}(\Phi_M(\omega \otimes \chi)) = \Phi_L(J_{P \cap M}^M(\omega \otimes \chi)) = \Phi_L(J_{P \cap M}^M(\omega) \otimes \chi) \\ \Phi_L(J_{P \cap M}^M(\omega)) \otimes \Phi_L(\chi) = \text{Res}_{\lambda_{LM}(\mathcal{H}^L)}^{\mathcal{H}^M}(\Phi_M(\sigma)) \otimes \Phi_{\text{nr}}(\chi) = \text{Res}_{\lambda_{LM}(\mathcal{H}^L)}^{\mathcal{H}^M}(\sigma \otimes \Phi_{\text{nr}}(\chi)).$$

Furthermore  $\Phi_M(\omega \otimes \chi)$  is an unramified twist of  $\sigma$ , because it lies in the same connected component of  $\text{Mod}(\mathcal{H}^M)$ . Hence

$$\Phi_M(\omega \otimes \chi) = \sigma \otimes \Phi_{\text{nr}}(\chi).$$

Then Condition 3.1.(i) guarantees that

$$(51) \quad \Phi_G(I_P^G(\omega \otimes \chi)) = \text{ind}_{\lambda_{MG}(\mathcal{H}^M)}^{\mathcal{H}^G}(\sigma \otimes \Phi_{\text{nr}}(\chi)) = \pi(Q, \sigma, \Phi_{\text{nr}}(\chi)).$$

Hence  $\Phi_G$  matches the finite length tempered elements of  $\text{Rep}(G)$  associated to  $(P, M, \omega)$  (via Theorem 2.2) with the finite dimensional tempered  $\mathcal{H}^G$ -modules associated to  $(Q, \sigma)$  (via Proposition 1.5). By Theorem 2.2  $I_P^G(\omega \otimes \chi)$  and  $I_P^G(\omega \otimes \chi')$  are isomorphic if and only if  $\chi' = w'\chi$  for some  $w' \in W'(T_\omega)$ . Analogously, Proposition 1.5 entails that  $\pi(Q, \sigma, t)$  and  $\pi(Q, \sigma, t')$  are isomorphic if and only if  $t' = g(t)$  for some  $g \in \mathcal{G}_{Q, \sigma}$ . From this and (49) we see that  $\Phi_{\text{nr}}$  induces a bijection

$$(52) \quad X_{\text{unr}}(M)/W'(T_\omega) \rightarrow T_{\text{un}}^Q/\mathcal{G}_{Q, \sigma} \cong X_{\text{unr}}(M)/\mathcal{G}'_{Q, \sigma}.$$

Comparing the outer sides of (52), we deduce that  $W'(T_\omega) = \mathcal{G}'_{Q, \sigma}$  as subgroups of  $\text{Aut}(X_{\text{nr}}(M))$ .

Now we come to the intertwining operators. Recall from (43) and (44) that  $I(w', \omega \otimes \chi)$  for  $w' \in W'(T_\omega)$  comes from a unitary element

$$(53) \quad \pi(w', \omega, \chi) : I_P^G(\omega \otimes \chi) \rightarrow I_P^G(\omega \otimes w'(\chi)).$$

For bookkeeping purposes we replace  $T^Q$  by  $X_{\text{nr}}(M)$  and  $\mathcal{G}_{Q, \sigma}$  by  $\mathcal{G}'_{Q, \sigma}$ , at the same time defining

$$\pi(Q, \sigma, \chi) := \pi(Q, \sigma, \Phi_{\text{nr}}(\chi)) \quad \text{and} \quad \pi(g', Q, \sigma, \chi) = \pi(g, Q, \sigma, \Phi_{\text{nr}}(\chi))$$

when  $g' \in \mathcal{G}'_{Q, \sigma}$  is a lift of  $g \in \mathcal{G}_{Q, \sigma}$ . In particular, for  $k \in X_{\text{nr}}(M, \sigma)$  the interwiner  $\pi(k, Q, \sigma, \chi)$  is the identity as map on the underlying vector spaces, it only changes  $\chi$  to  $k\chi$ . Then (20) says that the action of  $\mathcal{G}'_{Q, \sigma}$  in Proposition 1.5 and (28) comes from unitary intertwiners

$$(54) \quad \pi(g', Q, \sigma, \chi) \in \text{Hom}_{\mathcal{H}^G}(\pi(Q, \sigma, \chi), \pi(Q, \sigma, g'(\chi))).$$

Both (53) and (54) are unique up to scalars, because they depend algebraically on  $\chi$  and because for generic  $\chi \in X_{\text{unr}}(M)$  the involved representations are irreducible. Therefore, if  $w' = g'$  under  $W'(T_\omega) = \mathcal{G}'_{Q,\sigma}$ ,

$$(55) \quad \Phi_G(\pi(w', \omega, \chi)) = \pi(g', Q, \sigma, \Phi_{\text{nr}}(\chi))$$

up to a complex number of absolute value 1. To improve our situation, we simply replace  $\pi(w', \omega, \chi)$  by  $\Phi_G^{-1}(\pi(g', Q, \sigma, \Phi_{\text{nr}}(\chi)))$ . This normalization is harmless, because it does not change  $I(w', \omega \otimes \chi)$ . The advantage is that has become an actual equality.

Finally we are in a good position to compare the Schwartz algebras

$$(56) \quad \bigoplus_{(P,M,\omega)/\sim} (C^\infty(X_{\text{unr}}(M)) \otimes \text{End}_{\mathbb{C}}(I_P^G(V_\omega)^{K_s})^{W'(T_\omega)}) \quad \text{and} \\ \bigoplus_{(Q,\sigma)/\mathcal{G}} C^\infty(T_{\text{un}}^Q; \text{End}_{\mathbb{C}}(V_{Q,\sigma}))^{\mathcal{G}_{Q,\sigma}} = \bigoplus_{(Q,\sigma)/\mathcal{G}} C^\infty(X_{\text{unr}}(M); \text{End}_{\mathbb{C}}(V_{Q,\sigma}))^{\mathcal{G}'_{Q,\sigma}}.$$

To justify the equality in the second line, we note that a section of the algebra bundle over  $X_{\text{unr}}(M)$  is  $X_{\text{nr}}(M, \sigma)$ -invariant if and only if it descends to a section of the analogous algebra bundle over  $T_{\text{un}}^Q$ .

By the above constructions the  $\Phi_M$  provide a bijection between the indexing sets for the sums, so it suffices to compare

$$(57) \quad A_1 := C^\infty(X_{\text{unr}}(M); \text{End}_{\mathbb{C}}(I_P^G(V_\omega)^{K_s}))^{W'(T_\omega)} \quad \text{with} \\ A_2 := C^\infty(X_{\text{unr}}(M); \text{End}_{\mathbb{C}}(V_{Q,\sigma}))^{\mathcal{G}'_{Q,\sigma}}$$

when  $(P, M)$  corresponds to  $Q$  via Condition 3.2.(iv) and  $\Phi_M(\omega) = \sigma$ . The Morita equivalences  $\mathcal{S}(G, K_s)^s \sim_M \mathcal{S}(G)^s$  and  $\Phi_G$  send  $I_P^G(\omega \otimes \chi)^{K_s}$  to  $\pi(Q, \sigma, \chi)$  and by (55) this is compatible with the intertwining operators. Identifying  $W'(T_\omega)$  and  $\mathcal{G}'_{Q,\sigma}$ , we consider the following bimodules for  $A_1$  and  $A_2$ :

$$B_1 := C^\infty(X_{\text{unr}}(M); \text{Hom}_{\mathbb{C}}(I_P^G(V_\omega)^{K_s}, V_{Q,\sigma}))^{W'(T_\omega)}, \\ B_2 := C^\infty(X_{\text{unr}}(M); \text{Hom}_{\mathbb{C}}(V_{Q,\sigma}, I_P^G(V_\omega)^{K_s}))^{W'(T_\omega)}.$$

Here the  $W'(T_\omega)$ -actions are

$$\begin{aligned} (w' \cdot f_1)(w' \chi) &= \pi(w', \omega \otimes \chi) f_1(\chi) \pi(w', Q, \sigma, \chi)^{-1} & f_1 \in B_1, \\ (w' \cdot f_2)(w' \chi) &= \pi(w', Q, \sigma, \chi) f_2(\chi) \pi(w', \omega \otimes \chi)^{-1} & f_2 \in B_2. \end{aligned}$$

Notice that by the equality (55) these are honest group actions, not just up to some scalars. We claim that

$$(58) \quad B_1 \otimes_{A_1} B_2 \cong A_2 \quad \text{and} \quad B_2 \otimes_{A_2} B_1 \cong A_1$$

as bimodules over  $A_2$ , respectively  $A_1$ . Since all these algebras and modules are of finite rank over  $C^\infty(X_{\text{unr}}(M))^{W'(T_\omega)}$ , it suffices to check this locally, at any  $\chi \in X_{\text{unr}}(M)$ . Then the proof of (58) reduces to checking that

$$(59) \quad \text{Hom}_{\mathbb{C}}(I_P^G(V_\omega)^{K_s}, V_{Q,\sigma})^{W'(T_\omega)_\chi} \otimes_{\text{End}_{\mathbb{C}}(I_P^G(V_\omega)^{K_s})^{W'(T_\omega)_\chi}} \text{Hom}_{\mathbb{C}}(V_{Q,\sigma}, I_P^G(V_\omega)^{K_s})^{W'(T_\omega)_\chi} \\ \cong \text{End}_{\mathbb{C}}(V_{Q,\sigma})^{W'(T_\omega)_\chi},$$

and the other way round for  $B_2 \otimes_{A_2} B_1 \cong A_1$ .

By the uniqueness of  $\pi(w', Q, \sigma, \chi)$  up to scalars,  $w' \mapsto \pi(w', Q, \sigma, \chi)$  defines a projective representation of  $W'(T_\omega)_\chi$ . Let  $W''$  be a finite central extension of



$W'(T_\omega)_\chi$ , such that it lifts to a linear representation of  $W''$ . By (58) the map  $w' \mapsto \pi(w', \omega \otimes \chi)$  also lifts to a linear representation of  $W''$ . Then  $W''$  and  $W'(T_\omega)$  have the same invariants in the all involved modules, so we can rewrite (59) as

$$(60) \quad \begin{aligned} & \text{Hom}_{\mathbb{C}[W''']}(I_P^G(V_\omega)^{K_s}, V_{Q,\sigma}) \otimes_{\text{End}_{\mathbb{C}[W''']}(I_P^G(V_\omega)^{K_s})} \text{Hom}_{\mathbb{C}[W''']}(V_{Q,\sigma}, I_P^G(V_\omega)^{K_s}) \\ & \cong \text{End}_{\mathbb{C}[W''']}(V_{Q,\sigma}). \end{aligned}$$

This is a statement about finite dimensional representations of the finite group  $W''$ . One can verify (60) by reducing it to the case of irreducible  $W''$ -representations, where it is obvious.

This also proves (59) and (58), and shows that the algebras in (56) are Morita equivalent. Combining that with Theorem 2.2 and (28), we find the desired Morita equivalences of Schwartz algebras.

To prove that  $C_r(G)^\mathfrak{s}$  and  $C_r(\mathcal{R}, q) \rtimes \Gamma$  are Morita equivalent, we can use exactly the same argument. We only have to replace  $C^\infty$  by continuous functions everywhere, and to use Theorem 2.3 instead of Theorem 2.2.  $\square$

#### 4. HECKE ALGEBRAS FROM BUSHNELL–KUTZKO TYPES

Let  $L \subset G$  be a Levi subgroup and let  $\sigma \in \text{Irr}(L)$  be supercuspidal. Recall from [BuKu, §4] that a type for  $\mathfrak{s} = [L, \sigma]_G$  consists of a compact open subgroup  $J \subset G$ , and a  $\lambda \in \text{Irr}(J)$ , such that  $\text{Rep}(G)^\mathfrak{s}$  is precisely the category of smooth  $G$ -representations which are generated by their  $\lambda$ -isotypical subspace. To such a type one associates the algebra

$$\mathcal{H}(G, J, \lambda) = \text{End}_G(\text{ind}_J^G \lambda),$$

which (by definition) acts from the right on  $\text{ind}_J^G \lambda$ . Then there is a Morita equivalence

$$(61) \quad \begin{aligned} \Phi_G : \text{Rep}(G)^\mathfrak{s} & \rightarrow \text{Mod}(\mathcal{H}(G, J, \lambda)) \\ \pi & \mapsto \text{Hom}_J(\lambda, \pi) \cong \text{Hom}_G(\text{ind}_J^G \lambda, \pi). \end{aligned}$$

For a Levi subgroup  $M \subset G$  containing  $L$ , Bushnell and Kutzko [BuKu, §8] developed the notion that  $(J, \lambda)$  covers a  $[L, \sigma]_M$ -type  $(J_M, \lambda_M)$ . Roughly speaking, this means that  $J_M = J \cap M$ , that  $\lambda_M = \text{Res}_{J_M}^J \lambda$  and that  $\mathcal{H}(G, J, \lambda)$  contains invertible “strongly positive” elements. Under these conditions, writing  $\mathfrak{s}_M = [L, \sigma]_M$ , there is a Morita equivalence  $\Phi_M : \text{Rep}(M)^{\mathfrak{s}_M} \rightarrow \text{Mod}(\mathcal{H}(M, J_M, \lambda_M))$  as in (61), which is in several ways compatible with  $\Phi_G$ .

**Lemma 4.1.** *Suppose that  $(J, \lambda)$  is a cover of a  $[L, \sigma]_L$ -type  $(J_L, \lambda_L)$ . Then Condition 3.1 is fulfilled, with  $\mathcal{H}^M = \mathcal{H}(M, J_M, \lambda_M)$ .*

*Proof.* Let  $P$  and  $P'$  be as in the condition. By [BuKu, Proposition 8.5]  $(J_{M'}, \lambda_{M'})$  is a  $\mathfrak{s}_{M'}$ -type,  $(J_M, \lambda_M)$  is a  $\mathfrak{s}_M$ -type and the former covers the latter.

By [BuKu, Corollary 8.4] there exists a unique algebra monomorphism

$$t_{\overline{P} \cap M} : \mathcal{H}(M, J_M, \lambda_M) \rightarrow \mathcal{H}(M', J_{M'}, \lambda_{M'})$$

such that

$$\text{Res}_{t_{\overline{P} \cap M}(\mathcal{H}(M, J_M, \lambda_M))}^{\mathcal{H}(M', J_{M'}, \lambda_{M'})} \circ \Phi_{M'} = \Phi_M \circ \text{pr}_{\mathfrak{s}_M} \circ R_{\overline{P} \cap M}^{M'}.$$

Here  $R_{\overline{P} \cap M}^{M'}$  means the unnormalized parabolic restriction functor. To obtain the version with the normalized Jacquet functor  $J_{\overline{P} \cap M}^{M'}$ , we must adjust  $t_{\overline{P} \cap M}$  by the

square root of a modular character. This yields our  $\lambda_{MM'}$ . The uniqueness of  $\lambda_{MM'}$  and the transitivity of normalized Jacquet restriction entail that

$$\lambda_{M'M''} \circ \lambda_{MM'} = \lambda_{MM''} \quad \text{when } P \subset P' \subset P'' \subset G.$$

On general grounds  $\text{ind}_{\lambda_{MM'}(\mathcal{H}(M, J_M, \lambda_M))}^{\mathcal{H}(M', J_{M'}, \lambda_{M'})}$  is the left adjoint of  $\text{Res}_{\lambda_{MM'}(\mathcal{H}(M, J_M, \lambda_M))}^{\mathcal{H}(M', J_{M'}, \lambda_{M'})}$ . By Bernstein's second adjointness theorem  $I_{P \cap M'}^{M'} : \text{Rep}(M) \rightarrow \text{Rep}(M')$  is the left adjoint of  $J_{P \cap M'}^{M'}$ . Hence

$$I_{P \cap M'}^{M'} : \text{Rep}(M)^{\mathfrak{s}_M} \rightarrow \text{Rep}(M')^{\mathfrak{s}_{M'}}$$

is the left adjoint of  $\text{pr}_{\mathfrak{s}_M} \circ J_{P \cap M'}^{M'}$ . By the uniqueness of adjoints

$$\Phi_{M'} \circ I_{P \cap M'}^{M'} = \text{ind}_{\lambda_{MM'}(\mathcal{H}(M, J_M, \lambda_M))}^{\mathcal{H}(M', J_{M'}, \lambda_{M'})} \circ \Phi_M. \quad \square$$

Having checked Condition 3.1 in a general framework, we turn to more specific instances where Condition 3.2 holds. In most cases the intermediate algebras  $\mathcal{H}^M$  are not mentioned explicitly in the literature. One can obtain them by applying the same references to the group  $M$  instead of  $G$ . Using the canonical construction of  $\lambda_{MM'}$  as in the proof of Lemma 4.1, Condition 3.2.(ii) will be satisfied automatically in those cases. We will check the remaining conditions, mainly by providing relevant references.

### Iwahori–spherical representations.

This is the classical case. Let  $\mathfrak{o}_F$  be the ring of integers of the local non-archimedean field  $F$ , let  $\mathfrak{p}_F$  be its maximal ideal, and let  $k_F = \mathfrak{o}_F/\mathfrak{p}_F$  be the residue field. Choose an apartment  $\mathbb{A}$  of the Bruhat–Tits building of  $G$  and let  $L$  be the corresponding minimal  $F$ -Levi subgroup of  $G$ . Let  $I$  be an Iwahori subgroup of  $G$  associated to a chamber of  $\mathbb{A}$ . Let  $P_L$  be the parabolic subgroup of  $G$  with Levi factor  $L$ , such that the reduction of  $I$  modulo  $\mathfrak{p}_F$  is  $P_L(k_F)$ .

Borel [Bor] showed that the trivial representation of  $I$  is a  $\mathfrak{s}$ -type, where  $\mathfrak{s} = [L, \text{triv}_L]_G$ . Borel assumes that  $G$  is semisimple, but it is easy to generalize his arguments to reductive  $G$ .

By [IwMa, §3] there is an algebra isomorphism

$$(62) \quad C_c(I \backslash G/I) \cong \mathcal{H}(G, I, \text{triv}) \cong \mathcal{H}(X_*(L), R^\vee(G, L), X^*(L), R(G, L), \Delta, q_I),$$

where the basis  $\Delta$  is determined by  $P_L$  and  $q_{I, \alpha} = \text{vol}(Is_\alpha I)/\text{vol}(I)$  for a simple reflection  $s_\alpha$ . From [Bor, §3.1] one sees that Conditions 3.2.(iii) and (4) hold. Here  $\Gamma_M = 1$  for all  $M$ , so Condition 1.2 is vacuous.

Of course Proposition 3.3 was already known for irreducible Iwahori-spherical representations. Indeed, by [KaLu, Section 8] and [ABPS1, Theorem 15.1.(2) and Proposition 16.6] the bijection  $\text{Irr}(G)^{\mathfrak{s}} \rightarrow \text{Irr}(\mathcal{H}(G, I, \text{triv}))$  preserves temperedness and essential square-integrability. Moreover Theorem 3.4 has been proven for Schwartz algebras in [DeOp, Theorem 10.2]: (62) extends to an isomorphism of Fréchet  $*$ -algebras

$$\mathcal{S}(I \backslash G/I) \cong \mathcal{S}(X_*(L), R^\vee(G, L), X^*(L), R(G, L), \Delta, q_I).$$

### Principal series representations of split groups.

Suppose that  $G$  is  $F$ -split and let  $T = \mathcal{T}(F)$  be a maximal split torus of  $G$ . Fix a

smooth character  $\chi_{\mathfrak{s}} \in \text{Irr}(T)$  and put  $\mathfrak{s} = [T, \chi_{\mathfrak{s}}]_G$ , so that

$$X_{\text{nr}}(T) \rightarrow T_{\mathfrak{s}} : \chi \mapsto \chi \chi_{\mathfrak{s}}$$

is a homeomorphism. Notice that  $\chi$  restricted to the unique maximal compact subgroup  $T_{\text{cpt}}$  of  $T$  is a type for  $[T, \chi_{\mathfrak{s}}]_T$ . By [Roc, Lemma 6.2] there exist a root subsystem  $R_{\mathfrak{s}} \subset R^{\vee}(G, T)$  and a subgroup  $\mathfrak{R}_{\mathfrak{s}} \subset W_{\mathfrak{s}}$  such that  $W_{\mathfrak{s}} = W(R_{\mathfrak{s}}) \rtimes \mathfrak{R}_{\mathfrak{s}}$ .

**Theorem 4.2.** [Roc, Theorem 6.3]

*There exists a type  $(J, \lambda)$  for  $\mathfrak{s}$  and an algebra isomorphism*

$$\mathcal{H}(G, J, \lambda) \cong \mathcal{H}(T_{\mathfrak{s}}, R_{\mathfrak{s}}, q) \rtimes \mathfrak{R}_{\mathfrak{s}},$$

where  $q_{\alpha} = |k_F|$  for all  $\alpha \in R_{\mathfrak{s}}$ . Moreover  $(J, \lambda)$  is a cover of  $(T_{\text{cpt}}, \chi)$ .

Furthermore Conditions 3.2(iii) and (iv) hold by construction. If  $\mathbb{Q}R(M, T) = \mathbb{Q}Q^{\vee}$ , then  $X \rtimes W_Q \Gamma_Q \subset X_*(T) \rtimes W(M, T)$ , so by Remark 1.3 Condition 1.2 holds as well.

We note that for these Bernstein components Proposition 3.3.b was already proven in [Roc, Theorem 10.7], while Proposition 3.3.a follows from [DeOp, Theorem 10.1], using [Roc, Section 8].

### Level zero representations.

These are  $G$ -representations which contain non-zero vectors fixed by the pro-unipotent radical of a parahoric subgroup of  $G$ . Iwahori-spherical representations constitute the most basic example of this kind. A type  $(J, \lambda)$  for any Bernstein component  $\mathfrak{s}$  consisting of level zero representations was exhibited in [Mor1], while it was proven in [Mor2, Theorem 4.9] that it actually is a type. More precisely, by [Mor2, §3.8]  $(J, \lambda)$  is a cover of a type for the underlying supercuspidal Bernstein component of a Levi subgroup  $L$  of  $G$ .

By [Mor1, Theorem 7.12] (see also [Lus2])

$$(63) \quad \mathcal{H}(G, J, \lambda) \cong \mathcal{H}(\mathcal{R}, q) \rtimes \mathbb{C}[\Gamma, \mathfrak{z}_{\mathfrak{s}}]$$

for suitable  $\mathcal{R}, q$  and  $\Gamma$ . In all examples of level zero Bernstein blocks which have been worked out, the 2-cocycle  $\mathfrak{z}_{\mathfrak{s}}$  of  $\Gamma$  is trivial. But even if it were non-trivial, we could regard  $\mathcal{H}(G, J, \lambda)$  as a direct summand of  $\mathcal{H}(\mathcal{R}, q) \rtimes \Gamma^*$ , where  $\Gamma^*$  is a suitable finite central extension of  $\Gamma$ .

Conditions 3.2.(iii) and (iv) follow from the setup in [Mor1, §3.12–3.14] and [Mor2, §1.10], combined with the description of  $\mathcal{R}$  in [Mor1, Proposition 7.3]. The groups  $\Gamma_Q$  for  $\mathbb{Q}Q^{\vee} = \mathbb{Q}R(M, L)$  satisfy Condition 1.2 because are contained in  $X \rtimes W(M, S)$ , where  $W(M, S)$  is the Weyl group of  $M$  with respect to a maximal  $F$ -split torus  $S \subset L$ .

As in the above examples, there is previous work on temperedness also. It is claimed in [DeOp, Theorem 10.1] that Proposition 3.3.a holds here. For this one needs to know that (63) preserves the traces (maybe up to a positive factor) and the natural  $*$ -operations. The former follows from the support of the basis elements  $T_w$  of  $\mathcal{H}(G, J, \lambda)$  constructed in [Mor1] (only the unit element  $T_e$  is supported on  $J$ ). For a simple (affine) reflection  $s$ , both  $T_s$  and  $T_s^*$  have support  $J s J$ , so they differ only by a scalar factor. They also satisfy the same quadratic relation, so  $T_s^* = T_s$ . This implies that (63) is an isomorphism of  $*$ -algebras.

Knowing that [DeOp, Theorem 10.1] applies, and together with [ABPS1, Lemma 16.5] it also gives Proposition 3.3.b.

### Inner forms of $\mathrm{GL}_n(F)$ .

Let  $D$  be a division algebra with centre  $F$ . Every Levi subgroup of  $G = \mathrm{GL}_m(D)$  is of the form  $L = \prod_i \mathrm{GL}_{m_i}(D)^{e_i}$ , where  $\sum_i m_i e_i = m$ . Fix a supercuspidal  $\omega \in \mathrm{Irr}(L)$ , of the form  $\omega = \bigotimes_{i=1}^k \omega_i^{\otimes e_i}$ , where  $\omega_i \in \mathrm{Irr}(\mathrm{GL}_{m_i}(D))$  is supercuspidal and not inertially equivalent with  $\omega_j$  if  $i \neq j$ . Then  $T_{\mathfrak{s}} \cong \prod_{i=1}^k (\mathbb{C}^\times)^{e_i}$ ,  $R_{\mathfrak{s}}$  is of type  $\prod_{i=1}^k A_{e_i-1}$  and the stabilizer of  $\mathfrak{s} = [\omega, L]_G$  in  $W(G, L)$  is  $W(R_{\mathfrak{s}}) \cong \prod_{i=1}^k S_{e_i}$ .

### Theorem 4.3. [Séc, SéSt]

*There exists a type  $(J, \lambda)$  for  $\mathfrak{s}$ , which is a cover of a  $[\omega, L]_L$ -type. There exists a parameter function  $q_{\mathfrak{s}} : R_{\mathfrak{s}} \rightarrow q^{\mathbb{N}}$  such that there is an isomorphism of  $*$ -algebras*

$$\mathcal{H}(G, J, \lambda) \cong \mathcal{H}(X^*(T_{\mathfrak{s}}), R_{\mathfrak{s}}, X_*(T_{\mathfrak{s}}), R_{\mathfrak{s}}^\vee, q_{\mathfrak{s}}),$$

*where the right hand side is a tensor product of affine Hecke algebras of type  $\mathrm{GL}_e$  with  $e \leq m$ . Moreover this isomorphism sends the natural trace of  $\mathcal{H}(G, J, \lambda)$  to a positive multiple of the trace of the right hand side.*

We remark that the claims about the involution and the traces are not made explicit in [Séc, SéSt]. They can be deduced in the same way as for level zero representations, see above. With [DeOp, Theorem 10.1] that proves Proposition 3.3 for these groups.

Via the tensor product factorization Condition 3.2.(iii) reduces to the case of a supercuspidal representation  $\sigma^{\otimes e}$  of  $\mathrm{GL}_r(D)^e$ . There it is a consequence of the constructions involved in [Séc, Théorème 4.6], which entail that the same notion of positivity in real tori is used for  $(\mathrm{GL}_r(D)^e, \mathrm{GL}_1(D)^{re})$  and for  $\mathcal{H}(\mathrm{GL}_e, q)$ . Condition 3.2.(iv) is irrelevant because all the groups  $\Gamma_M$  are trivial.

For the Schwartz algebras of these groups Theorem 3.4 can be found in [ABPS2, Theorem 6.2]. The proof over there is similar but simpler, because not all complications from Section 3 arise.

### Inner forms of $\mathrm{SL}_n(F)$ .

Let  $G$  be the kernel of the reduced norm map  $\mathrm{GL}_m(D) \rightarrow F^\times$ . It is an inner form of  $\mathrm{GL}_n(F)$ , and every inner form looks like this. It was shown in [ABPS1] that for every inertial equivalence class  $\mathfrak{s}$ ,  $\mathcal{H}(G)^{\mathfrak{s}}$  is Morita equivalent with an algebra which is closely related to affine Hecke algebras of type  $\mathrm{GL}_e$  (yet is of a more general kind). It is not known whether there exists a  $\mathfrak{s}$ -type for every  $\mathfrak{s}$ , but in any case the constructions in [ABPS1] are derived from the work of Sécherre and Stevens on inner forms of  $\mathrm{GL}_n(F)$ , so types are not far away. Condition 3.1.(i) [ABPS2, Theorem 1.5.b], the maps  $\lambda_{MM'}$  are simply inclusions, and Condition 3.1.(ii) follows from that by the uniqueness of adjoints.

Condition 3.2 does not hold precisely for the algebras  $\mathcal{H}^M$  obtained in this setting (in fact the Plancherel isomorphism for these  $\mathcal{H}^M$  has not been worked out), so we cannot apply Proposition 3.3 or Theorem 3.4. Nevertheless the conclusions of these results hold, see [ABPS2].

Let us summarize the conclusions from this section.

**Corollary 4.4.** *Let  $\mathfrak{s} = [L, \sigma]_G$  be an inertial equivalence class of the kind discussed in this paragraph (principal series of split group, level zero, inner form of  $\mathrm{GL}_n(F)$  or  $\mathrm{SL}_n(F)$ ). Then  $\mathcal{S}(G)^\mathfrak{s}$  is Morita equivalent to the Schwartz completion of an extended affine Hecke algebra and  $C_r^*(G)^\mathfrak{s}$  is Morita equivalent to the  $C^*$ -completion of the same extended affine Hecke algebra.*

*Proof.* Except for the last case, this follows by applying Theorem 3.4. We just checked that all its assumptions are fulfilled. For the inner forms of  $\mathrm{SL}_n(F)$ , [ABPS2, Theorem 6.4] gives the result in the case of Schwartz algebras. Like the proof of Theorem 3.4, the method in [ABPS2, §6.2] also works for the  $C^*$ -algebras, with minor modifications.  $\square$

## 5. HECKE ALGEBRAS FROM BERNSTEIN'S PROGENERATORS

We return to the notations from Sections 2 and 3. Let  $\mathfrak{s} = [L, \sigma]_G$  be any inertial equivalence class for  $G$ . Bernstein [BeRu, §3] constructed a projective generator  $\Pi_\mathfrak{s}$  for the category  $\mathrm{Rep}(G)^\mathfrak{s}$ . By [Ren, VI.10.1], for any Levi subgroup  $M \subset G$  containing  $L$ :

$$\Pi_{\mathfrak{s}_M} = I_{P_L \cap M}^M(\Pi_{\mathfrak{s}_L}),$$

and this is a progenerator of  $\mathrm{Rep}(M)^{\mathfrak{s}_M}$ . In other words, the map

$$\Phi_M : V \mapsto \mathrm{Hom}_M(I_{P_L \cap M}^M(\Pi_{\mathfrak{s}_L}), V)$$

is an equivalence between  $\mathrm{Rep}(M)^{\mathfrak{s}_M}$  and the category of right modules of  $\mathrm{End}_M(I_{P_L \cap M}^M \Pi_{\mathfrak{s}_L})$ . For  $P_L \subset P = MU_P \subset G$  we put

$$\mathcal{H}^M = \mathrm{End}_M(I_{P_L \cap M}^M \Pi_{\mathfrak{s}_L})^{\mathrm{op}} = \mathrm{End}_M(\Pi_{\mathfrak{s}_M})^{\mathrm{op}}.$$

Then  $\Phi_M$  provides an equivalence of categories  $\mathrm{Rep}(M)^{\mathfrak{s}_M} \rightarrow \mathrm{Mod}(\mathcal{H}^M)$ .

**Lemma 5.1.** *In the above setting Condition 3.1 is fulfilled.*

*Proof.* The functoriality of normalized parabolic induction gives natural injections

$$\lambda_{MM'} : \mathcal{H}^M \rightarrow \mathcal{H}^{M'} \quad \text{for } P \subset P' \subset G.$$

By naturality the  $\lambda_{MM'}$  satisfy Condition 3.1.(iii). By Bernstein's second adjointness theorem, for  $V' \in \mathrm{Rep}(M')^{\mathfrak{s}_{M'}}$ :

$$\begin{aligned} \Phi_M(J_{P \cap M'}^{M'} V') &= \mathrm{Hom}_M(I_{P_L \cap M}^M \Pi_{\mathfrak{s}_L}, J_{P \cap M'}^{M'} V') \\ &\cong \mathrm{Hom}_{M'}(I_{P \cap M'}^{M'} I_{P_L \cap M}^M \Pi_{\mathfrak{s}_L}, V') \cong \mathrm{Hom}_{M'}(I_{P_L \cap M'}^{M'} \Pi_{\mathfrak{s}_L}, V') = \Phi_{M'}(V') \end{aligned}$$

as  $\mathcal{H}^M$ -modules (via  $\lambda_{MM'}$ ). This establishes the first commutative diagram in Condition 3.1. As in the proof of Lemma 4.1, the second commutative diagram follows from that by invoking the uniqueness of left adjoints.  $\square$

In the remainder of this section we assume that  $G$  is:

- a symplectic group, non necessarily split,
- or a special orthogonal group, non necessarily split,
- or an inner form of  $\mathrm{GL}_n(F)$ .

Besides the discussion of inner forms of  $\mathrm{GL}_n(F)$  in the previous section, we point out that types for Bernstein components of symplectic or special orthogonal groups have been constructed in [MiSt]. However, as far as we know the Hecke algebras associated to these types are in only few cases known explicitly.

For the groups listed above, Heiermann has subjected  $(\mathcal{H}^G)^{\text{op}} = \text{End}_G(I_{P_L}^G \Pi_{\mathfrak{s}_L})$  to a deep study. In [Hei1] he proved that it is an extended affine Hecke algebra with positive parameters. The constructions in [Hei1, §5] are such that every  $\text{End}_M(I_{P_L \cap M}^M \Pi_{\mathfrak{s}_L})$  arises as a parabolic subalgebra. For Condition 3.2.(iii) see [Hei2, §3]. It served as a step towards Proposition 3.3 for these groups [Hei2, Théorème 5].

By [Hei1, Proposition 1.15] the groups  $W_Q \Gamma_Q$  are always contained in  $W(\tilde{R}_Q)$  where  $\tilde{R}_Q \subset \mathbb{Q}R_Q$  is a larger root system. In view of Remark 1.3, Condition 3.2.(iv) holds.

In fact, a more precise description of the root data and the groups  $\Gamma_M$  is available. By [Hei1, 1.13] the root datum underlying the affine Hecke algebra  $\text{End}_G(I_{P_L}^G \Pi_{\mathfrak{s}_L})$  is a tensor product of root data of four types:  $\text{GL}_n$ ,  $\text{Sp}_{2n}$ ,  $\text{SO}_{2n+1}$  and  $\text{SO}_{2n}$ . The groups  $\Gamma_M$  are described in [Hei1, 1.15], but unfortunately some elements were overlooked, for the complete picture we refer to [Gol]. The only nontrivial  $\Gamma_M$  come from the type  $D$  factors, it can happen that for a root datum of type  $(\text{SO}_{2n})^e$  we have Weyl groups

$$(64) \quad W_M \cong W(D_n)^e, \quad W_M \Gamma_M \cong W(D_{ne}) \cap W(B_n)^e.$$

Then  $|\Gamma_M| = 2^{e-1}$ . In the above setting, Theorem 3.4 says:

**Theorem 5.2.** *Let  $G$  be a symplectic group or a special orthogonal group (not necessarily split), or an inner form of  $\text{GL}_n(F)$ . Let  $\mathfrak{s}$  be any inertial equivalence class for  $G$ .*

*Then  $\mathcal{S}(G)^{\mathfrak{s}}$  is Morita equivalent with the Schwartz completion of an extended affine Hecke algebra. The underlying root datum is a tensor product of root data of type  $\text{GL}_n$ ,  $\text{Sp}_{2n}$ ,  $\text{SO}_{2n+1}$  and  $\text{SO}_{2n}$ , and the group  $\Gamma$  is a direct product of groups as (64). Furthermore  $C_r^*(G)^{\mathfrak{s}}$  is Morita equivalent with the  $C^*$ -completion of that extended affine Hecke algebra.*

Theorem 5.2 was one of the motivations for the author to write a paper about the K-theory of  $C^*$ -completions of (extended) affine Hecke algebras [Sol2]. It enables us to show that the K-groups of the reduced  $C^*$ -algebras of the above groups are free.

**Theorem 5.3.** *Let  $G$  be as in Theorem 5.2. Then  $K_*(C_r^*(G))$  is a free abelian group. It is countably infinite, unless  $G = 1$ .*

*Proof.* Recall the Bernstein decomposition from (31):

$$C_r^*(G) \cong \prod_{\mathfrak{s} \in \mathfrak{B}(G)} C_r^*(G)^{\mathfrak{s}}.$$

Since topological K-theory is a continuous functor on the category of Banach algebras, it commutes with direct sums. This reduces the theorem to one factor  $C_r^*(G)^{\mathfrak{s}}$ . By Morita invariance and Theorem 5.2, it suffices to show that the K-theory of the  $C^*$ -completion of an extended affine Hecke algebra as in Theorem 5.2 is a finitely generated free abelian group. It was checked in [Sol2, (62)] that the Künneth theorem for topological K-theory [Sch] applies to such algebras. Thus we only need to prove the result when the underlying root datum is of type  $\text{GL}_n$ ,  $\text{Sp}_{2n}$  or  $\text{SO}_{2n+1}$  and  $\Gamma$  is trivial, and when the root datum is of type  $(\text{SO}_{2n})^e$  and  $\Gamma$  is as in (64). These K-groups were computed in [Sol2], see respectively Theorem 3.2, Theorem 3.3, (128), and Proposition 3.5. They are free abelian and have finite rank.  $\square$

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